Modern Foundations of Light Transport Simulation

Christian Lessig

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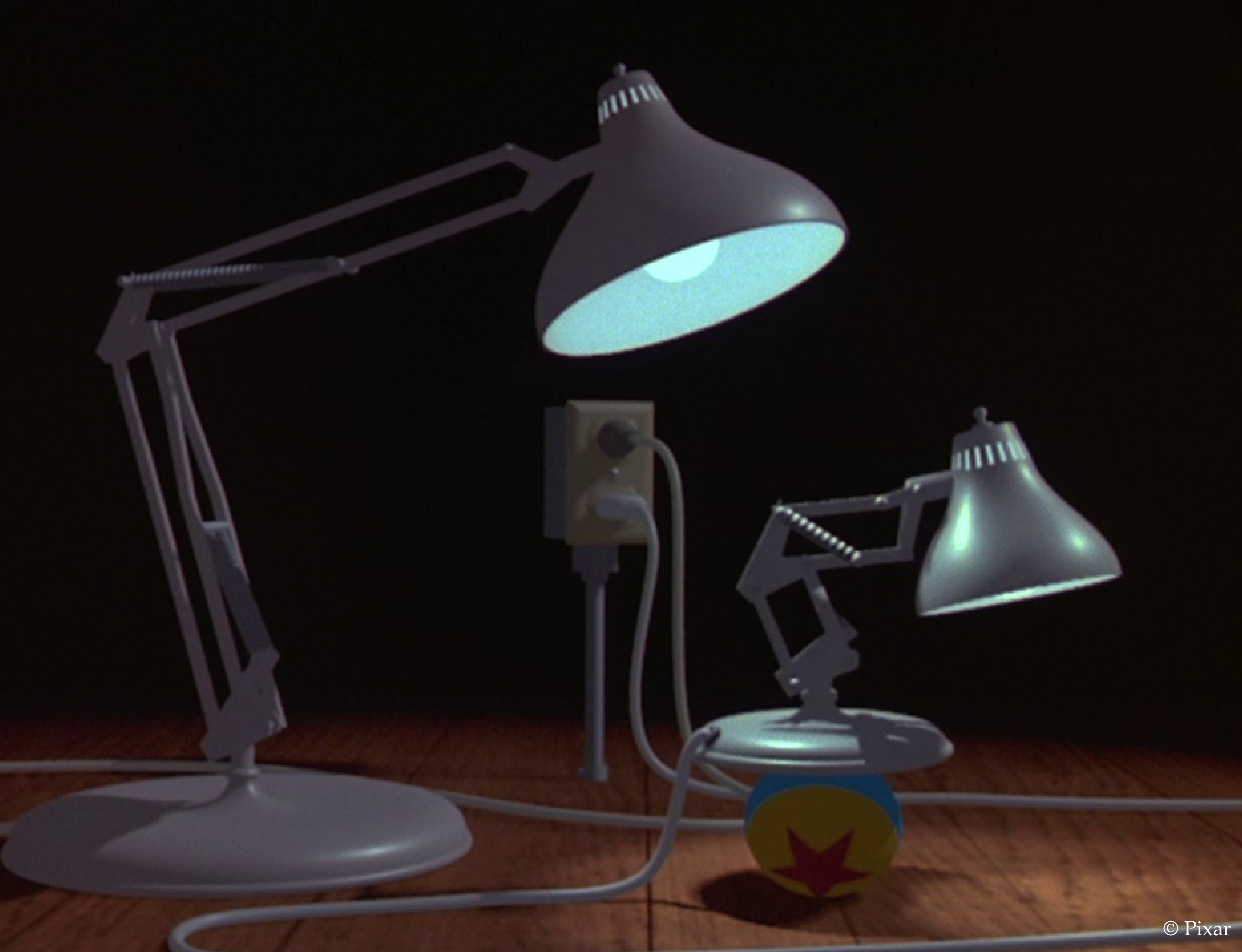
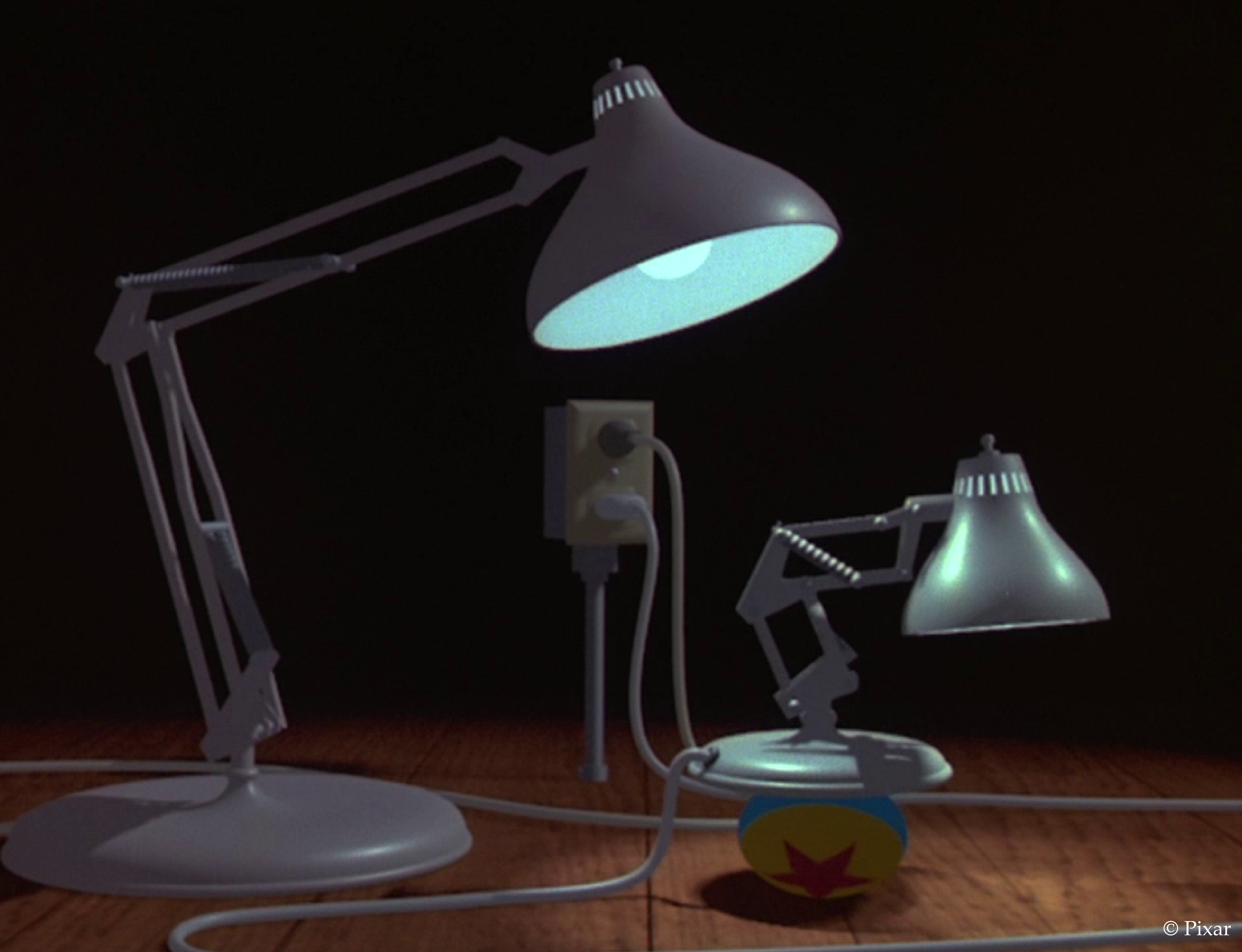
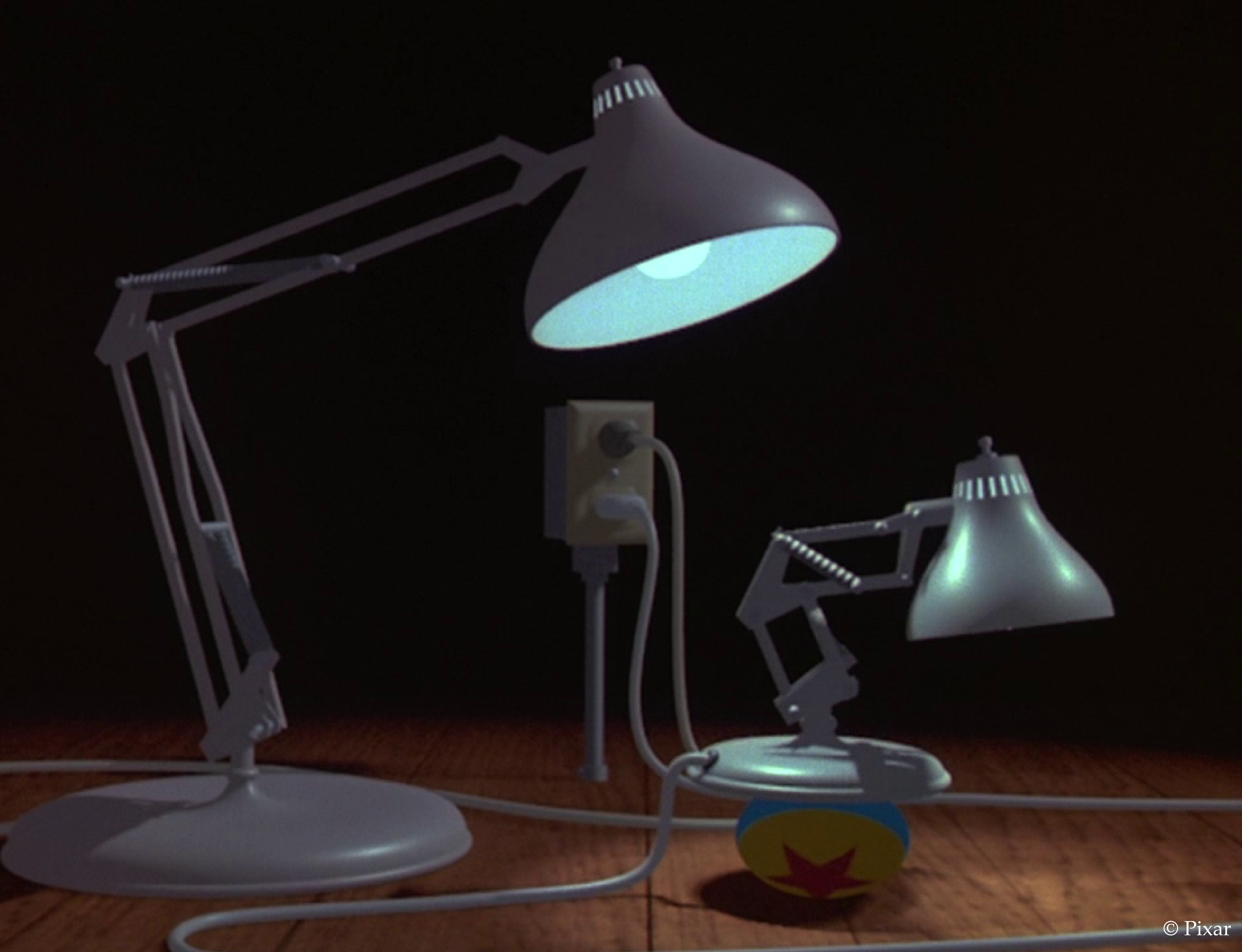
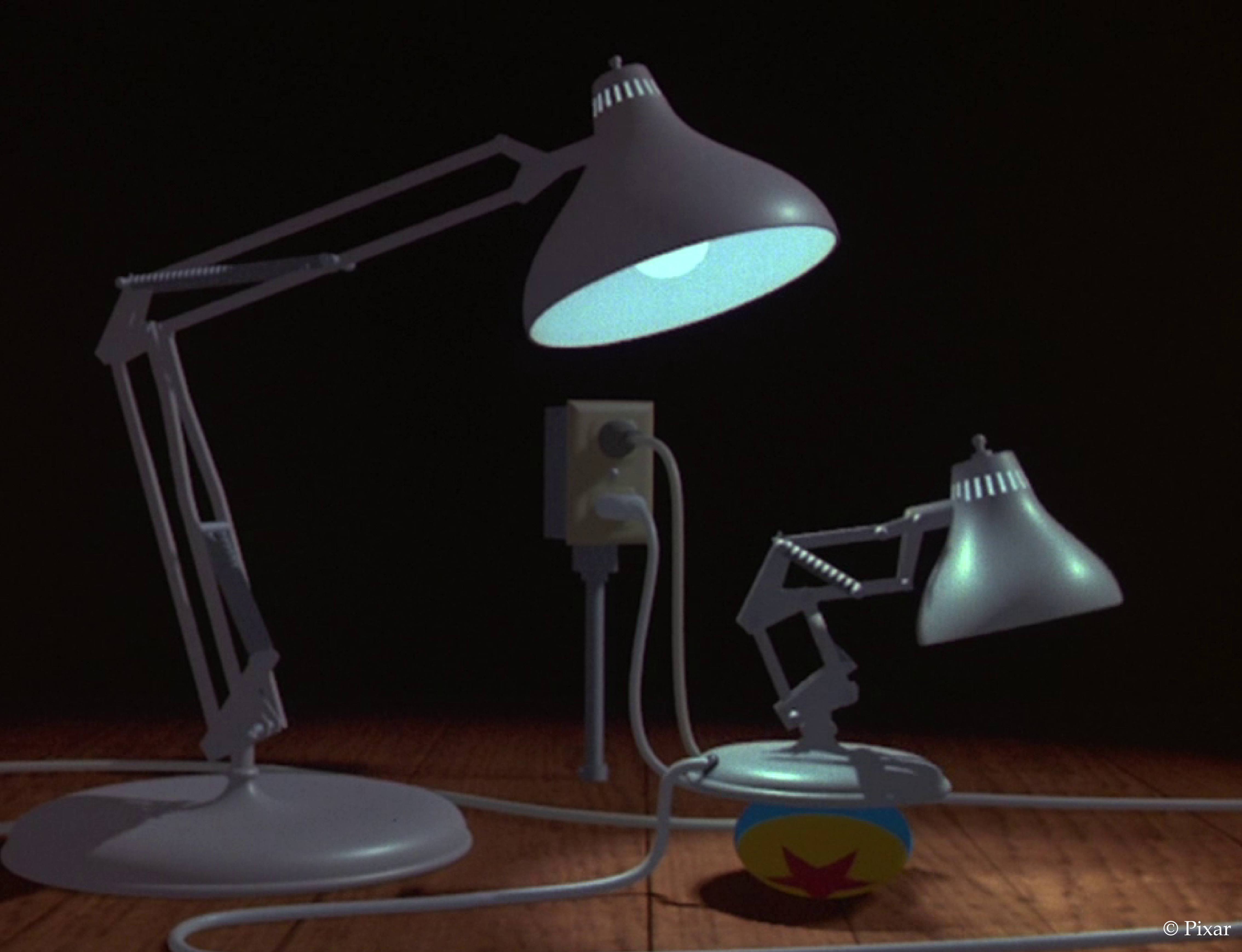
"Theoretical photometry constitutes a case of 'arrested development', and has remained basically unchanged since 1760 while the rest of physics has swept triumphantly ahead. In recent years, however, the increasing needs [. . .] have made the absurdly antiquated concepts of traditional photometric theory more and more untenable."[1]

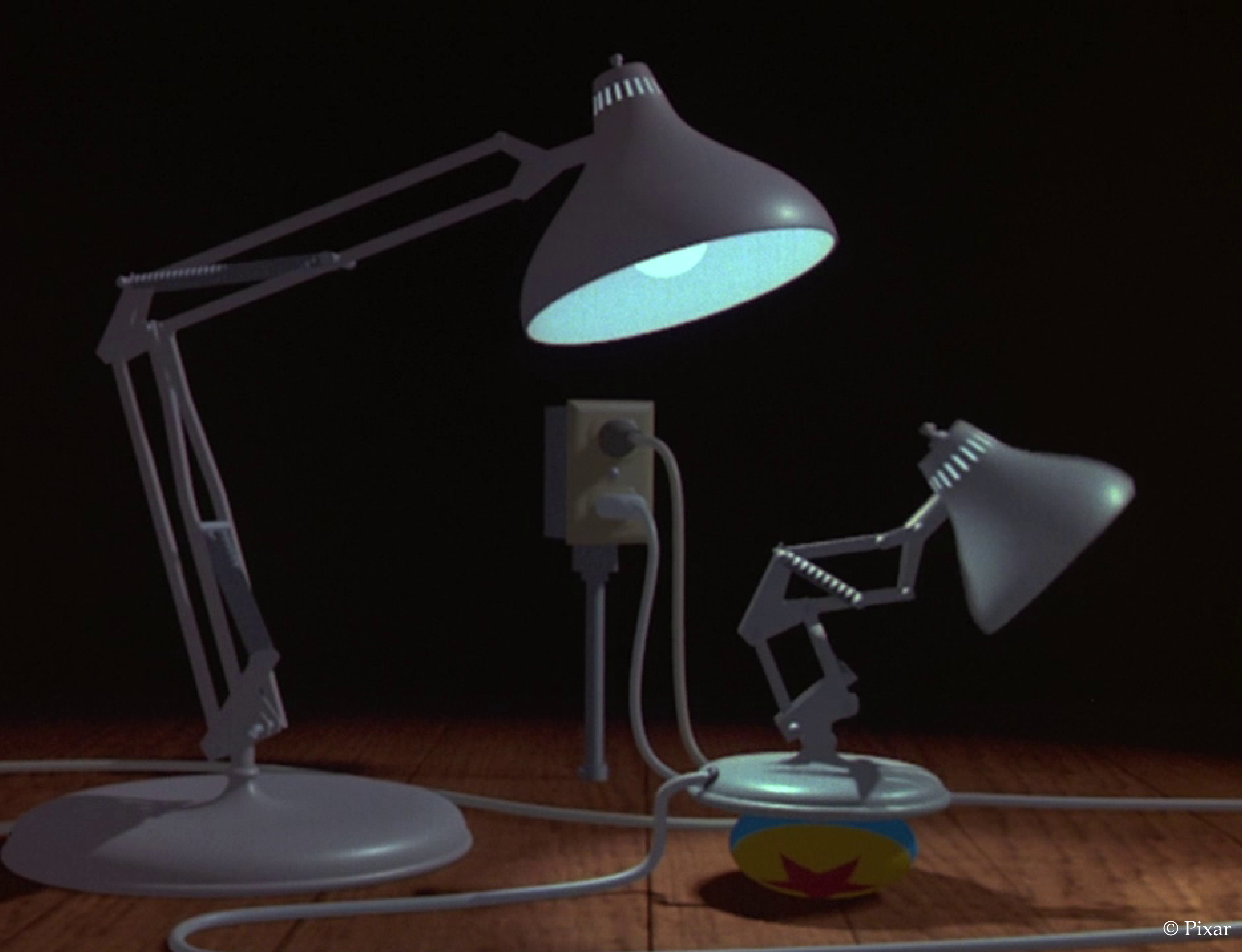
1. Gershun, A. *The Light Field*, Translated by P. Moon, G. Timoshenko, Originally published in Russian (Moscow 1936). Journal of Mathematics and Physics 18 (1939): 51-151, from the translators preface.

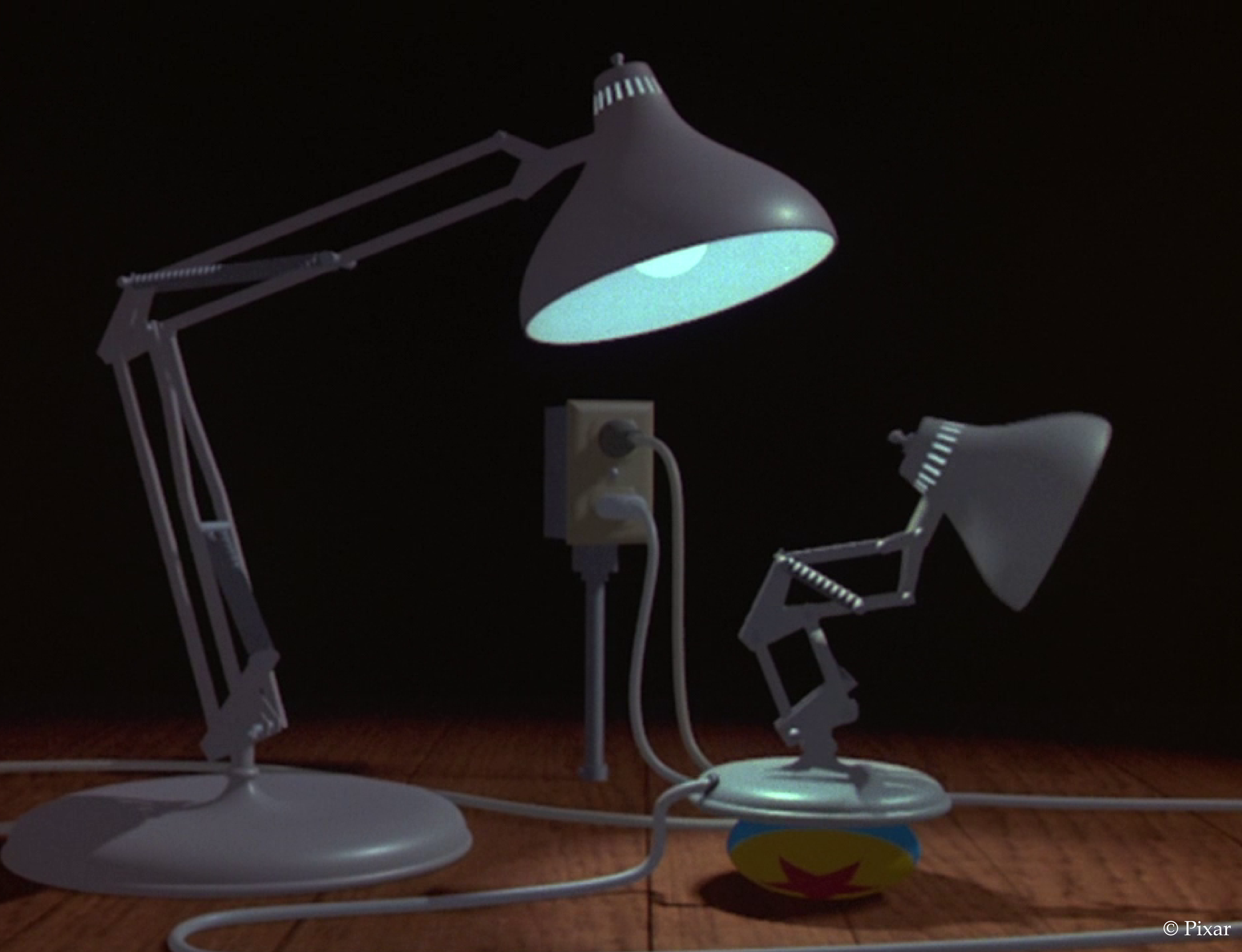












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configuration space Q

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{\epsilon}\nabla\times\\ \frac{1}{\mu}\nabla\times & 0\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$$

electromagnetic theory

configuration space Q

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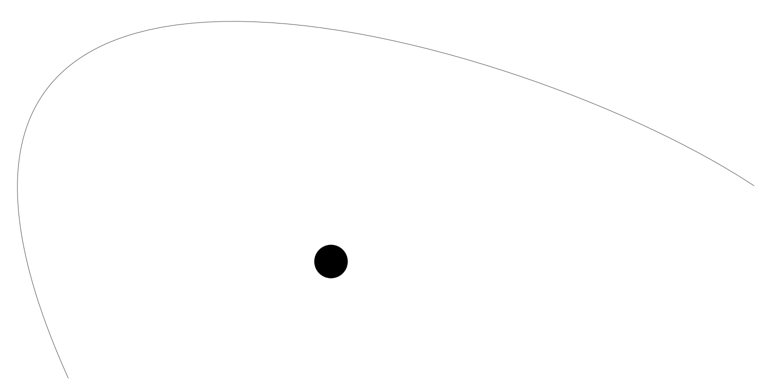
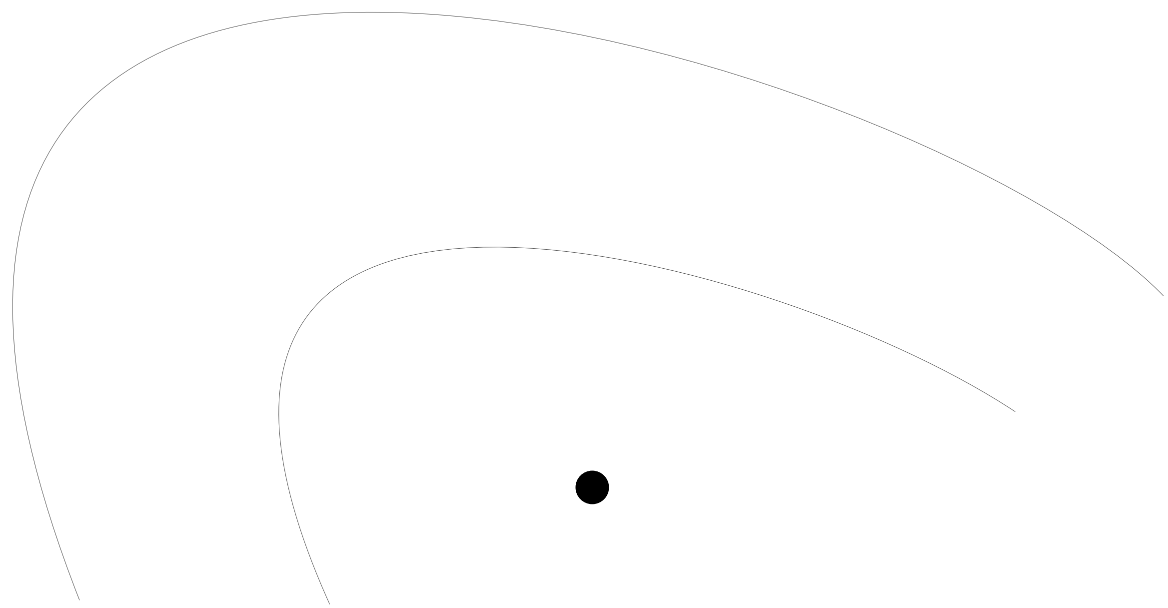
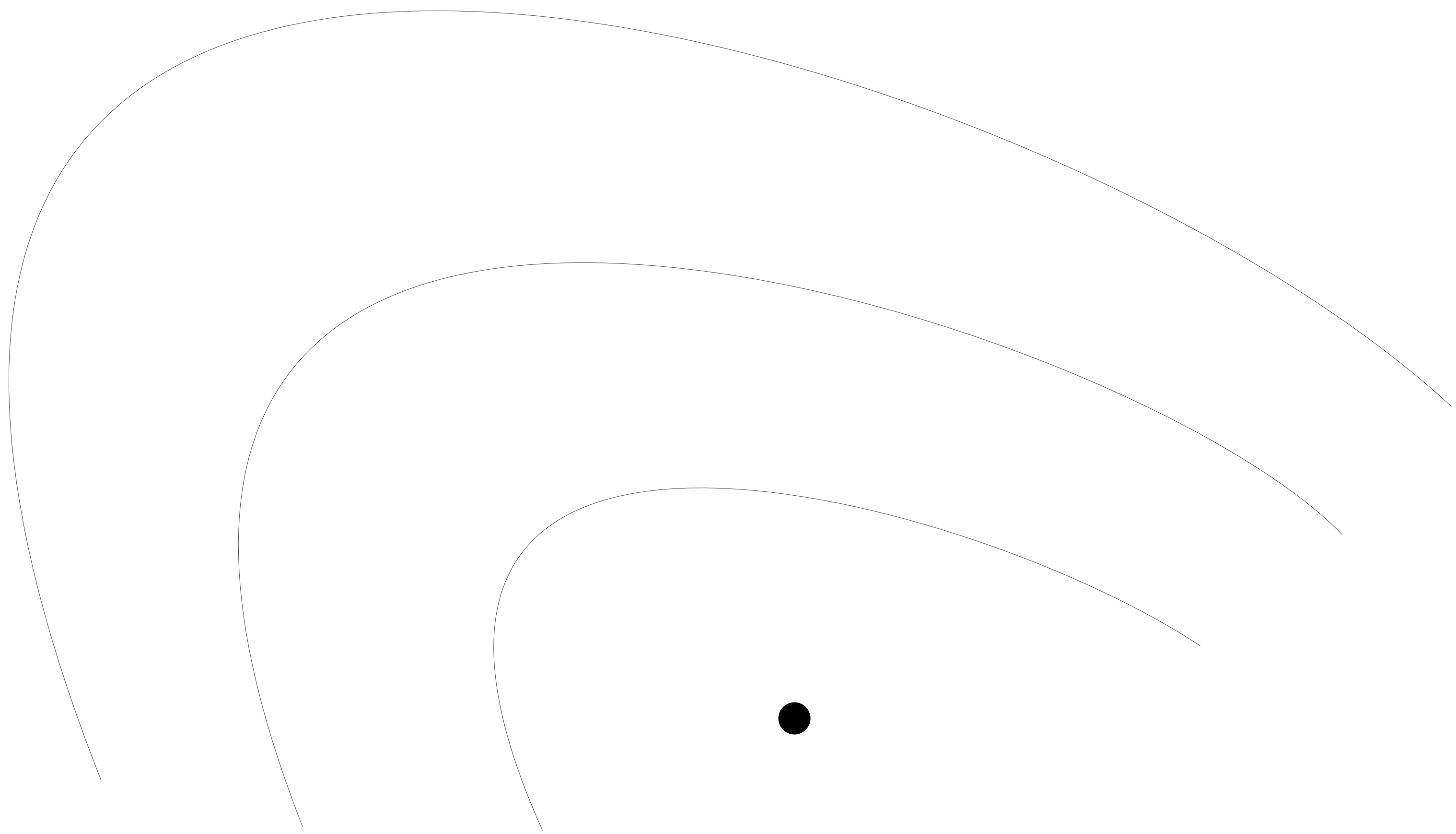
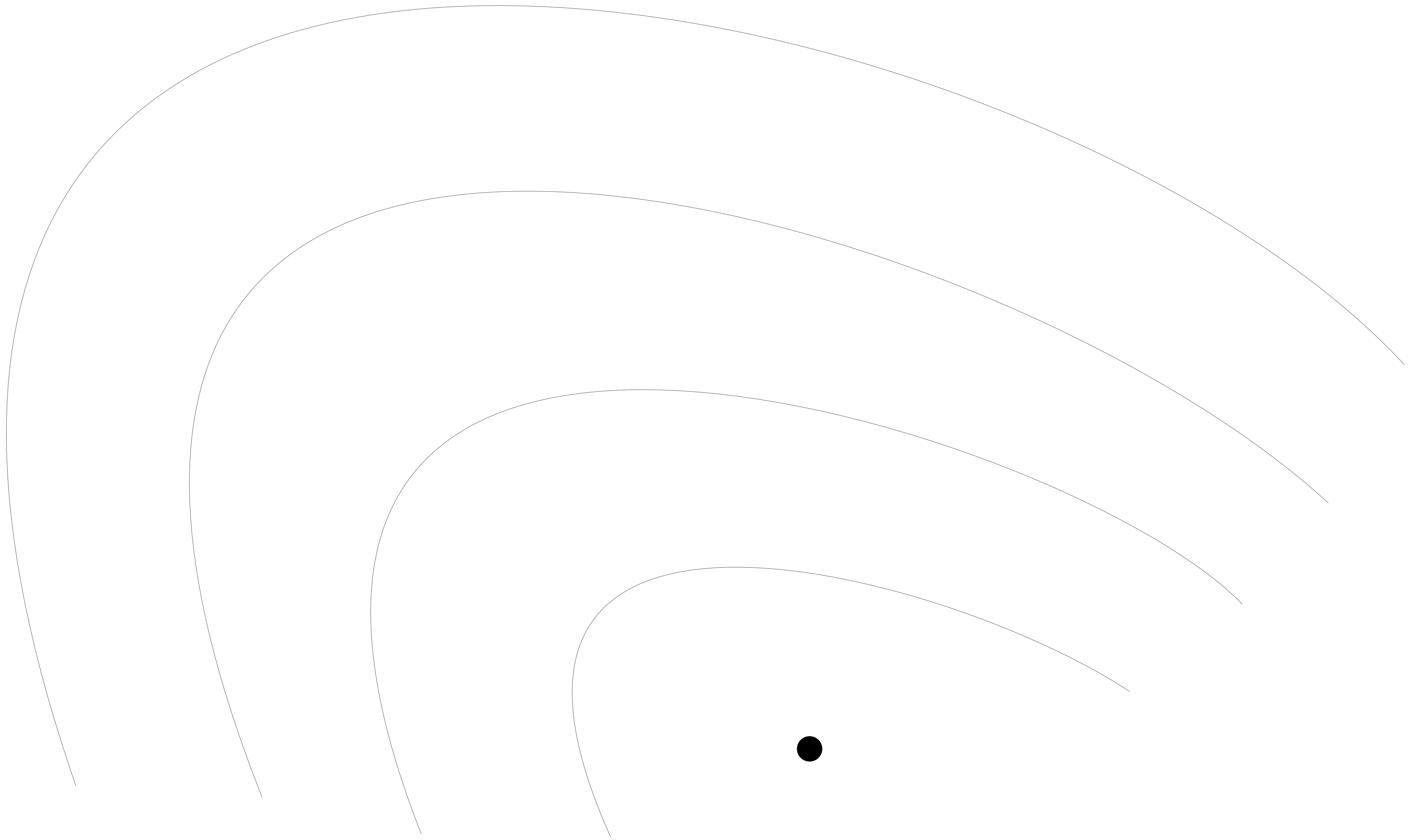
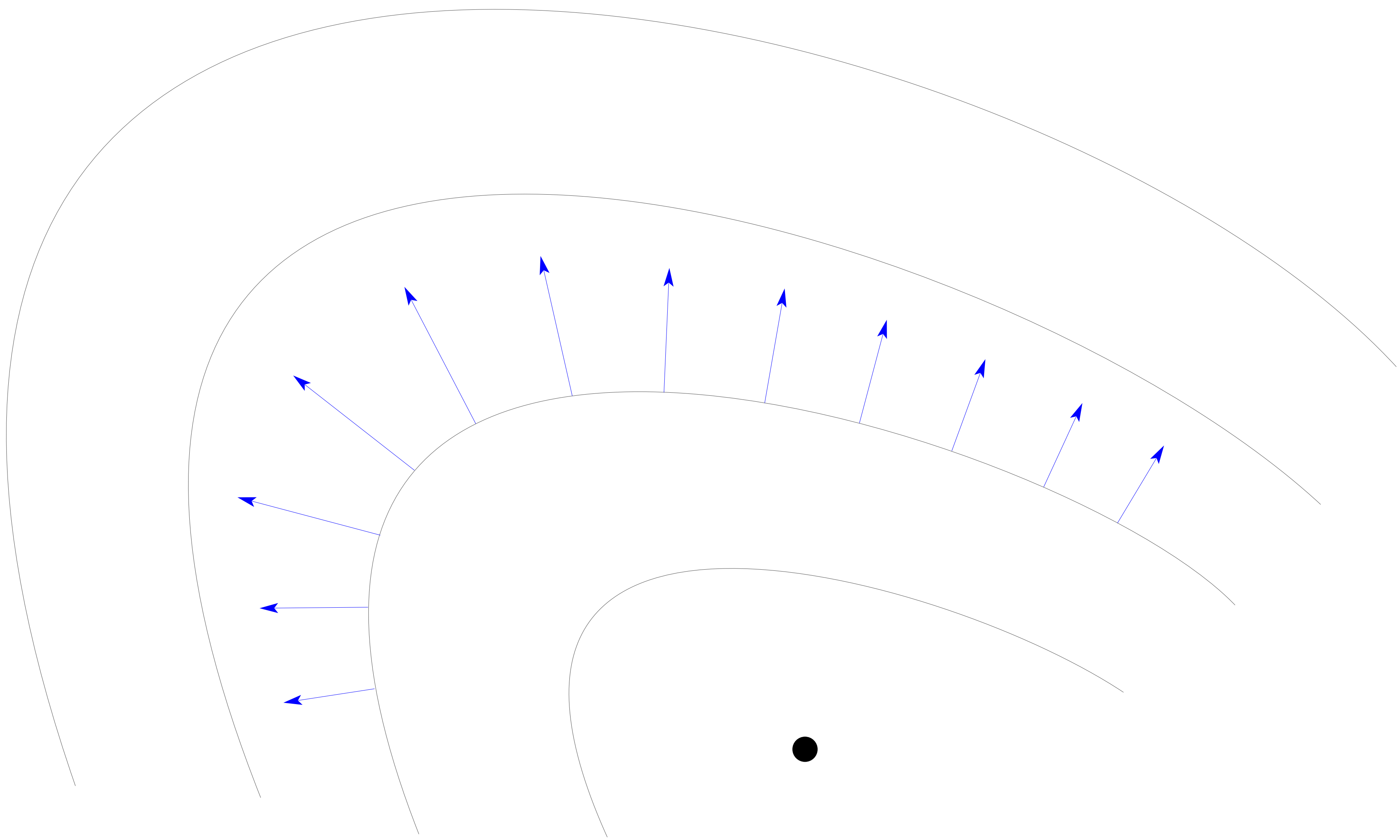
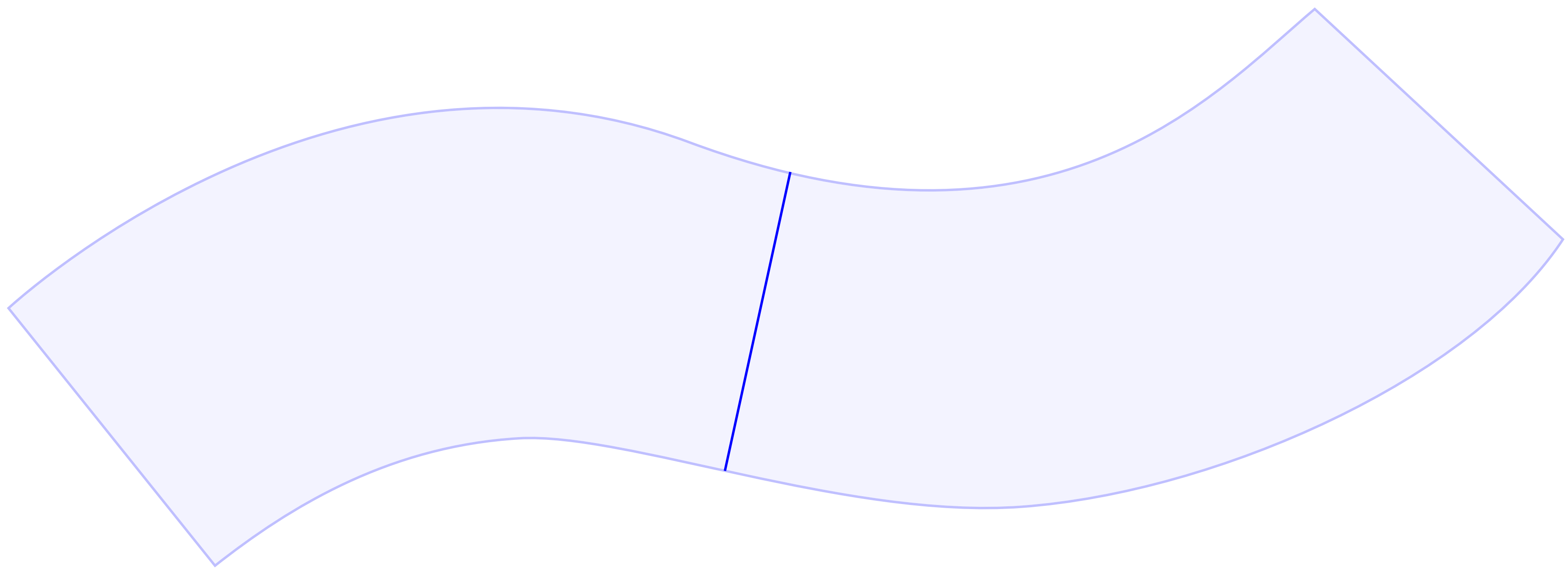


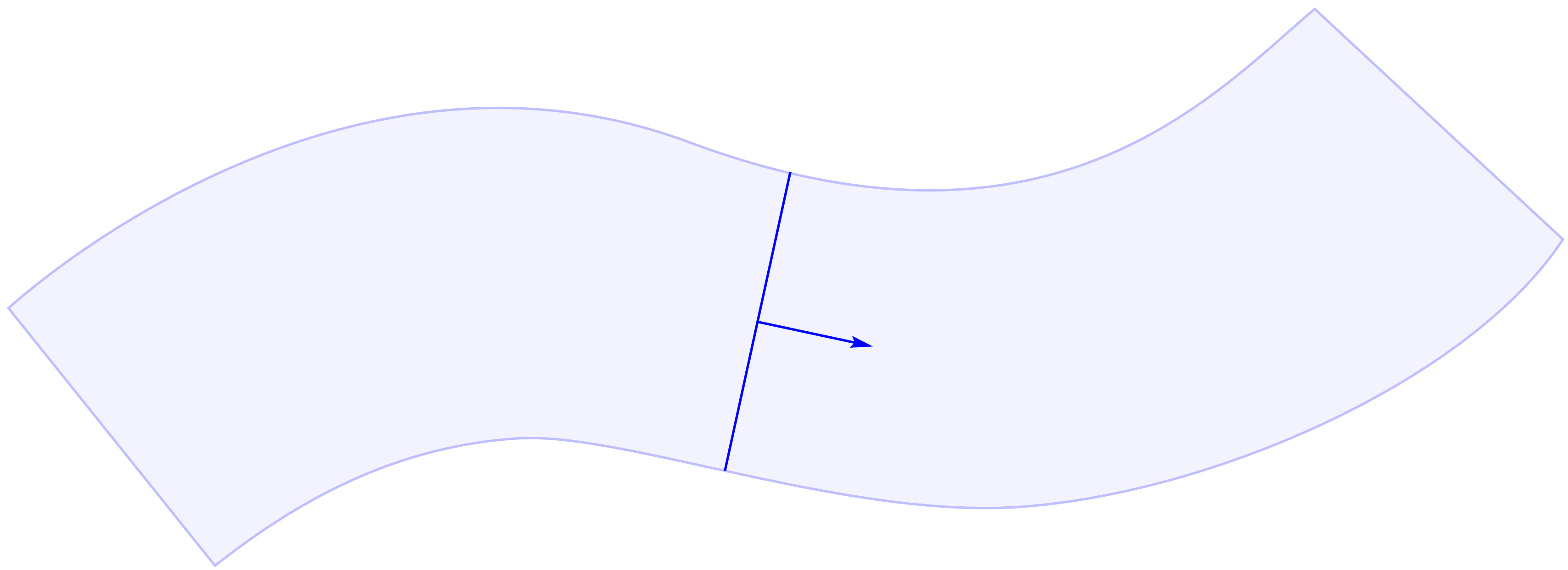


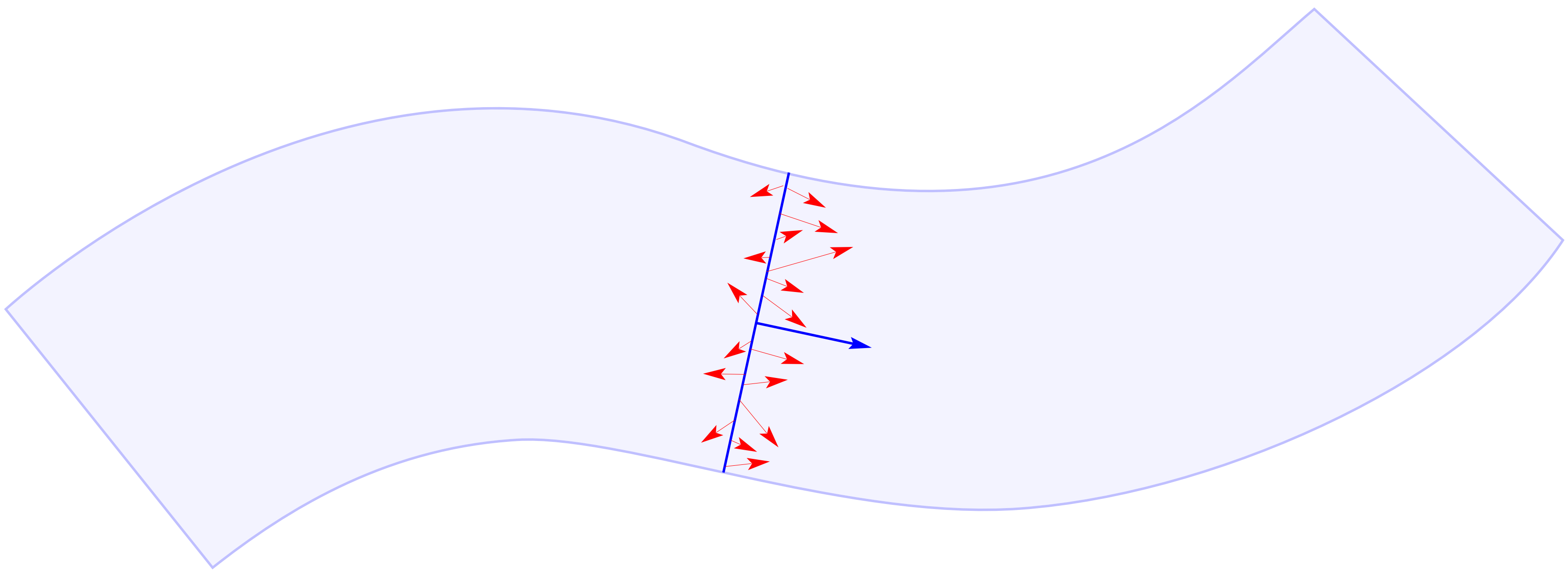


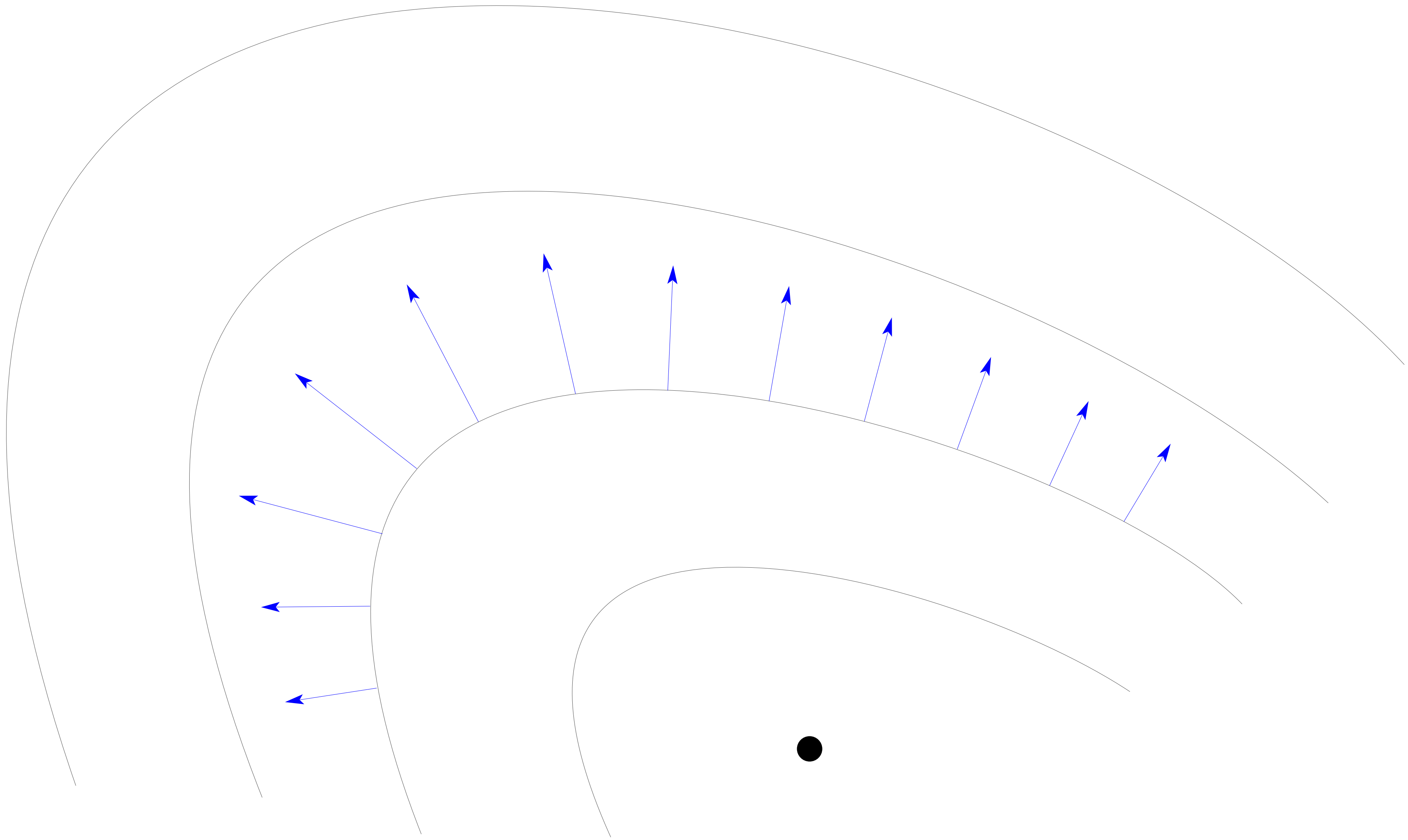


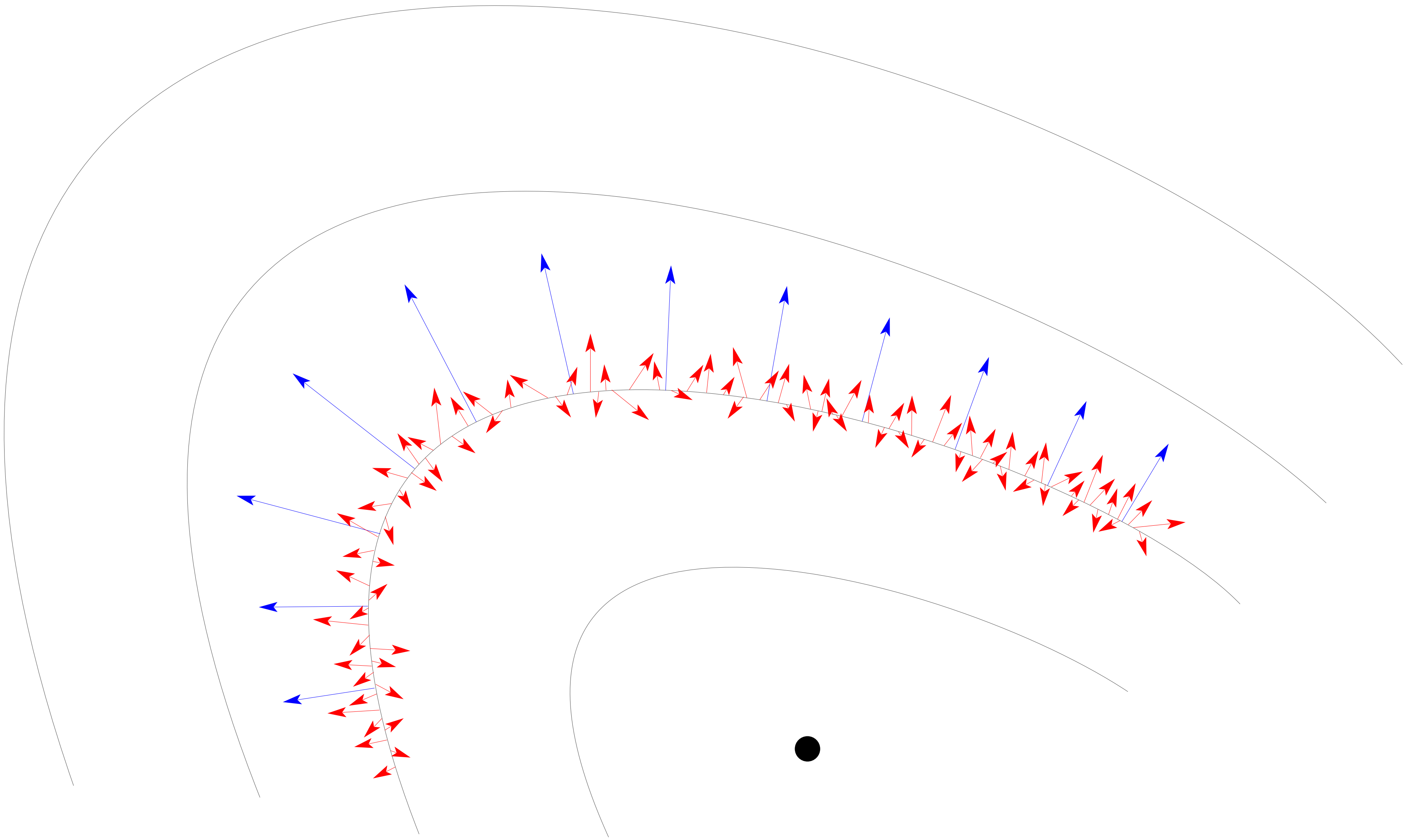


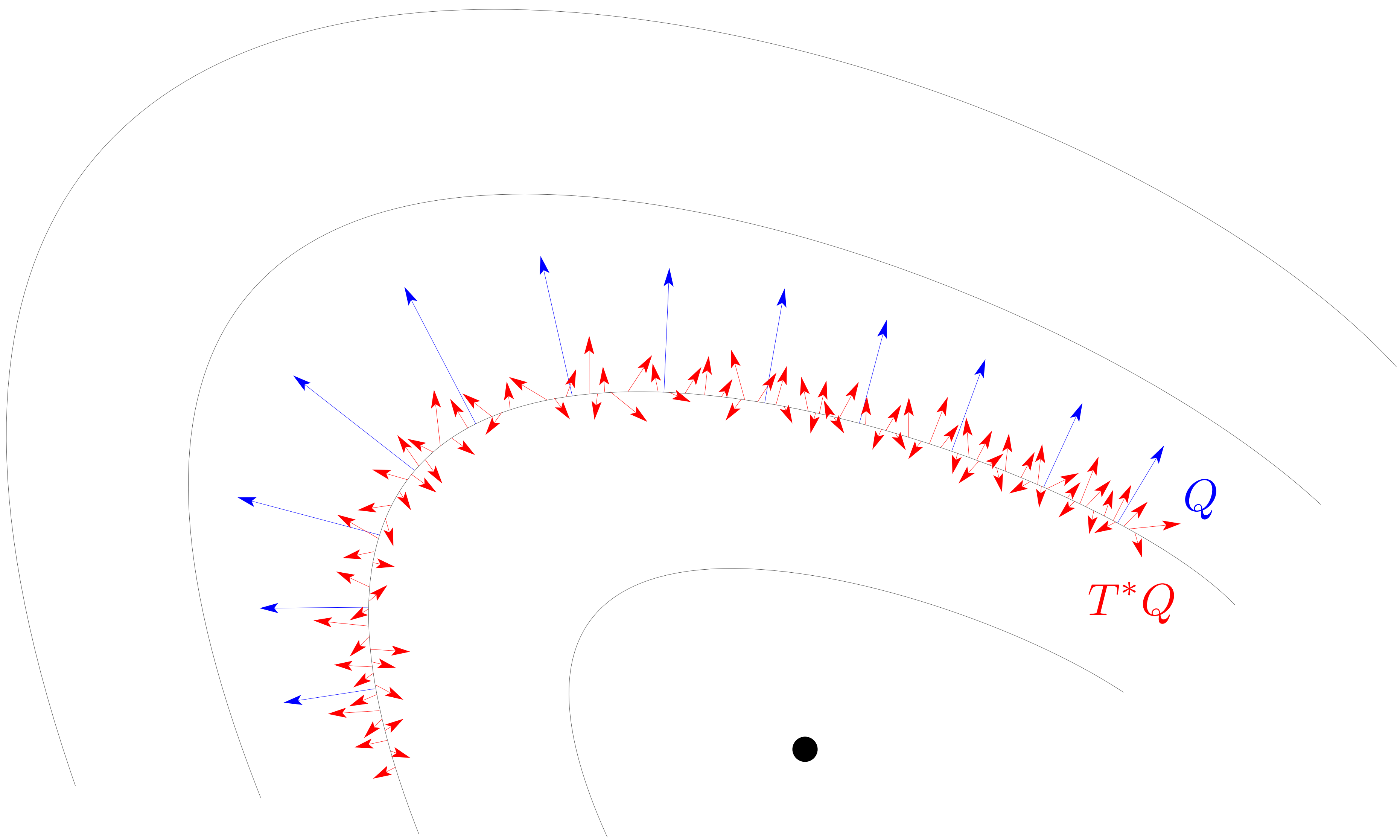


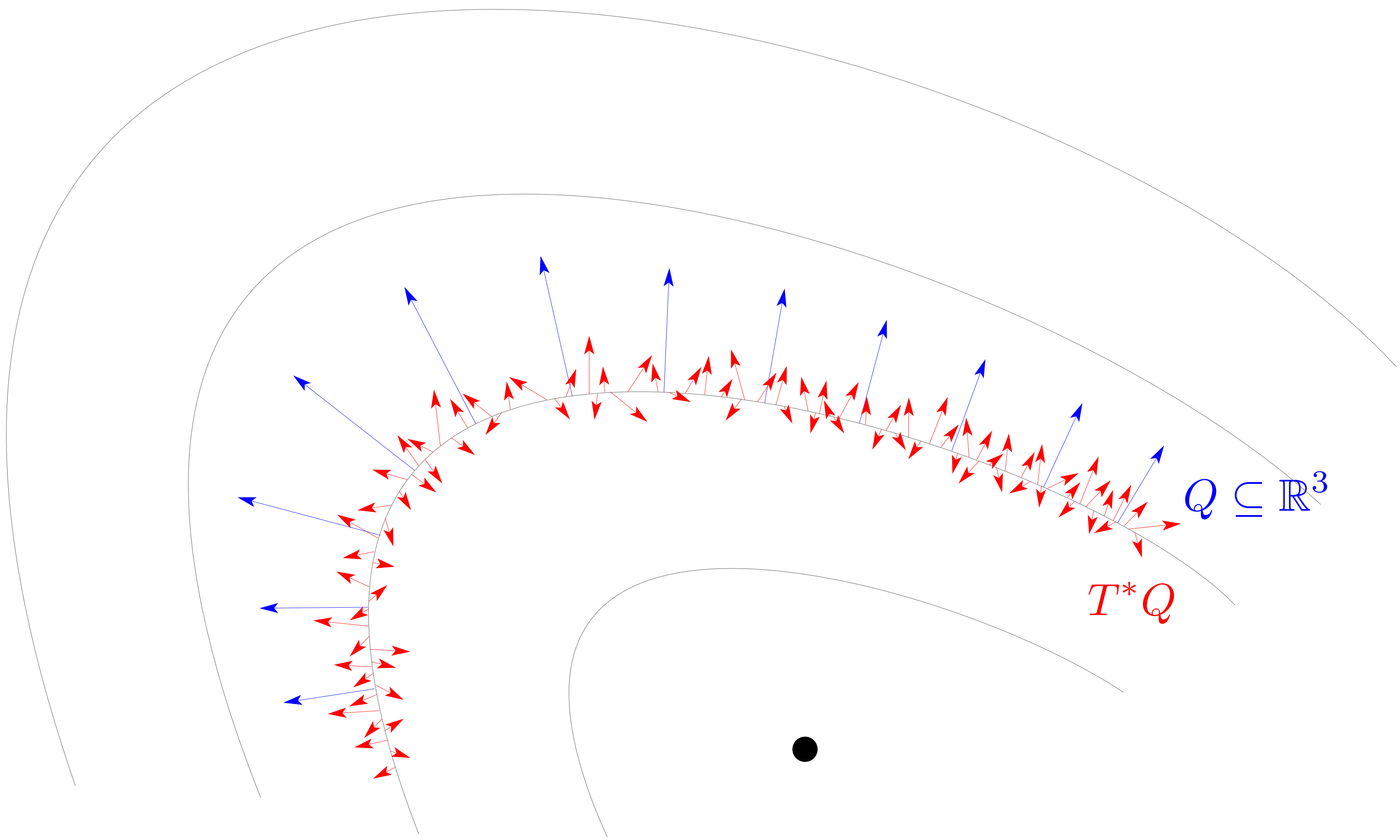




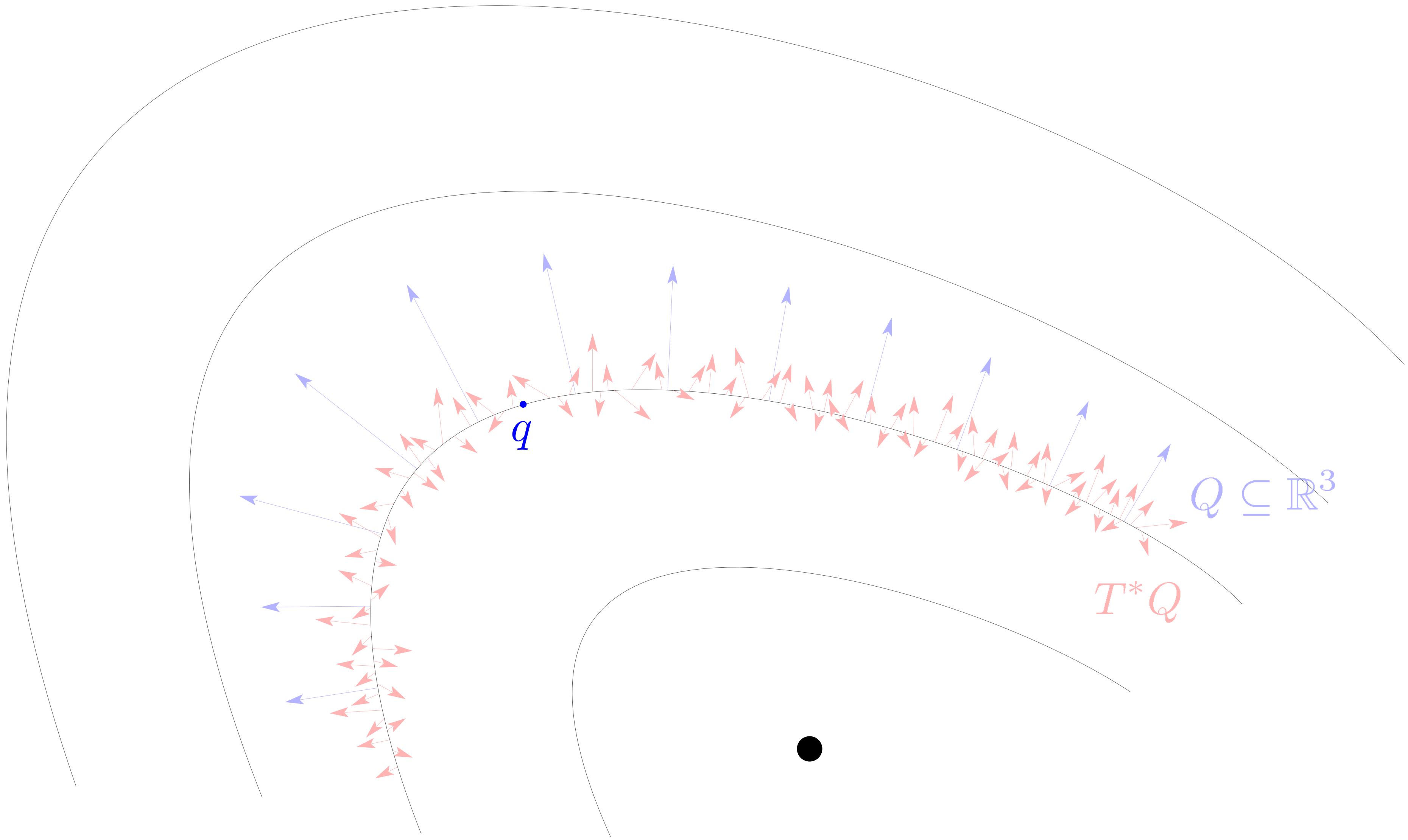





Q
T^*Q



$Q \subseteq \mathbb{R}^3$
T^*Q



q
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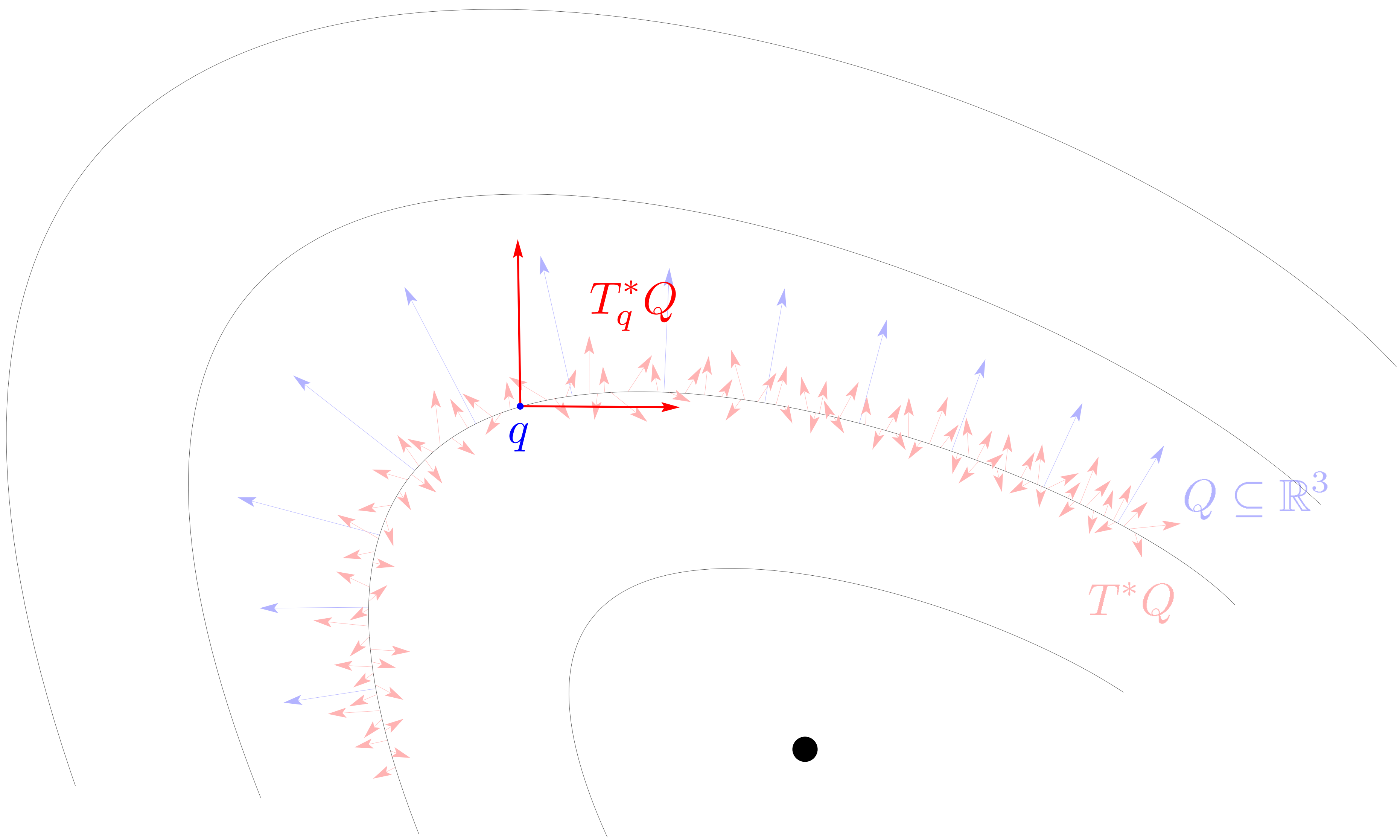
T^*Q


T_q^*Q
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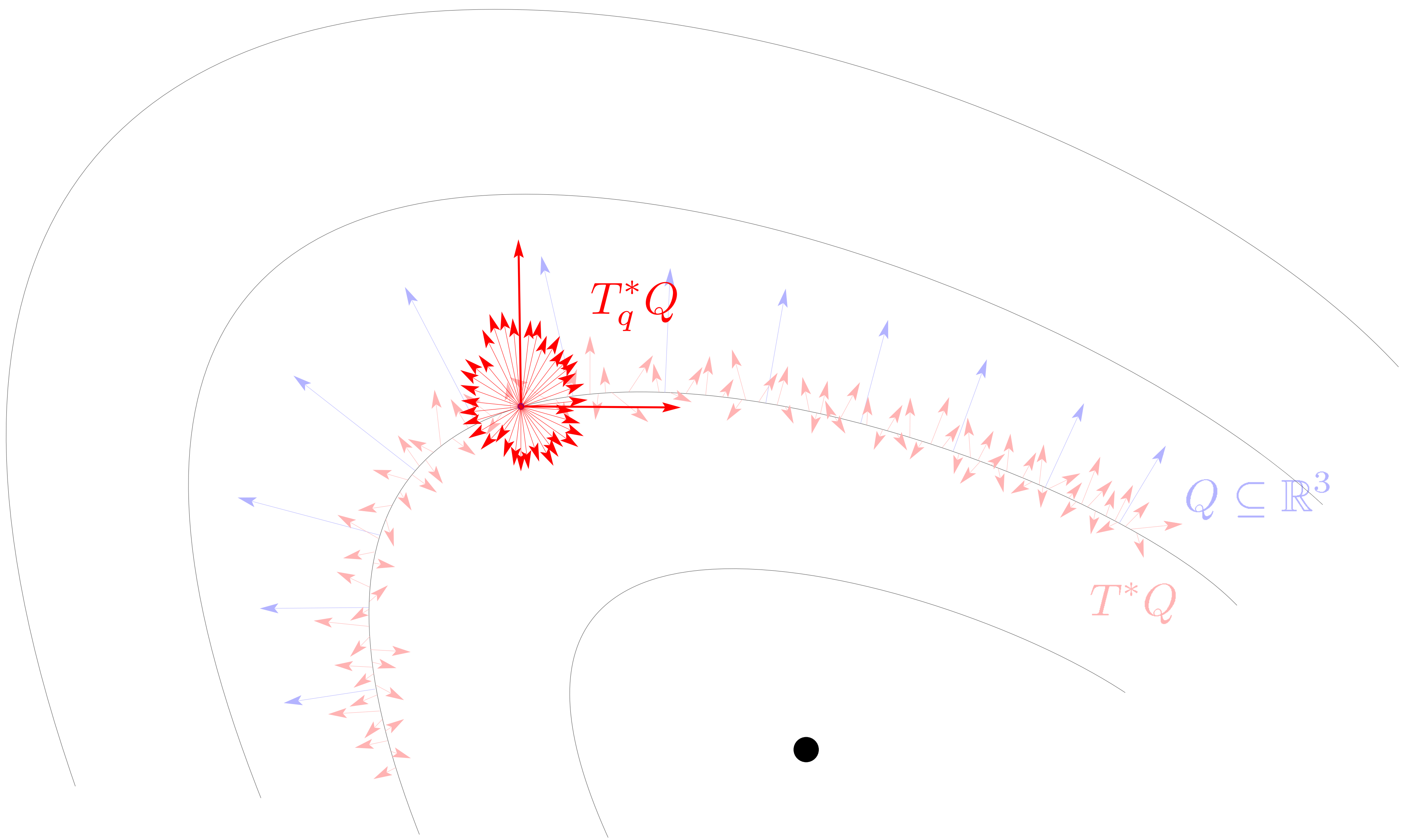

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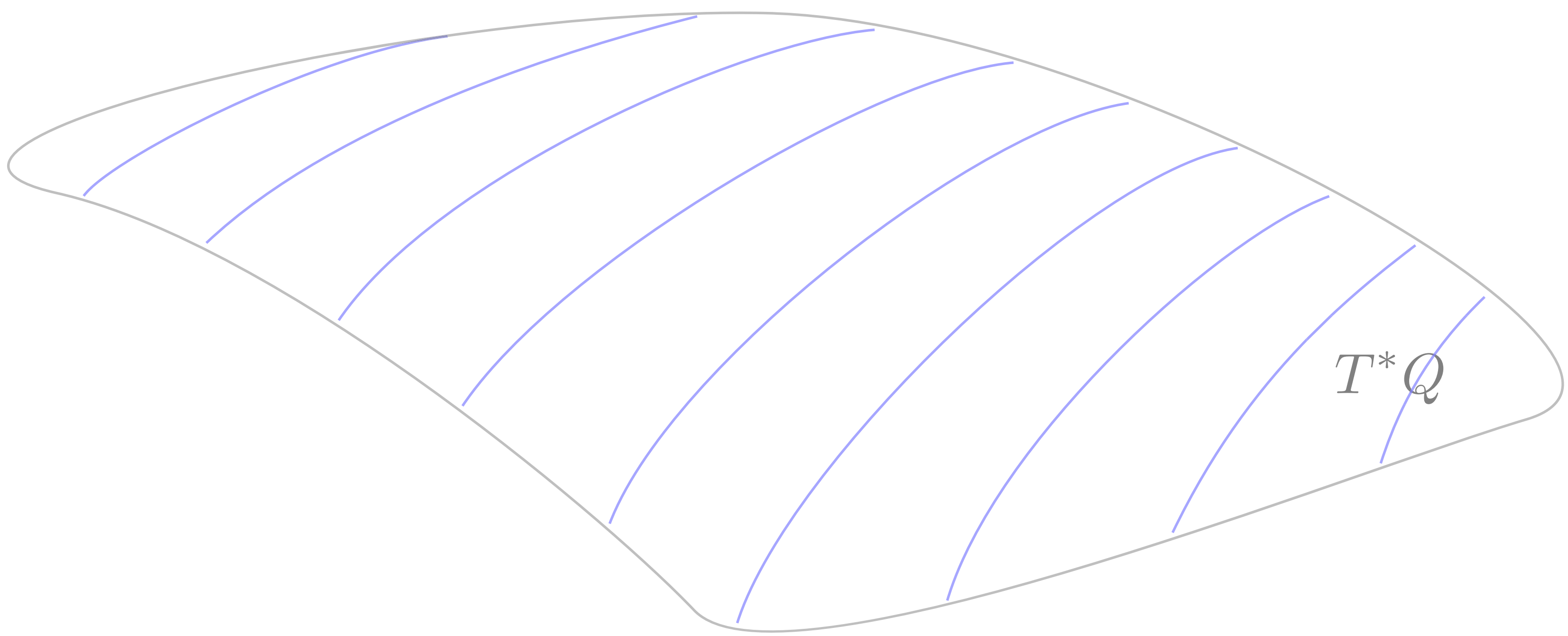
T^*Q

electromagnetic theory

configuration space Q

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$$

↓ microlocal analysis (Wigner transform)

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$$-\dot{W}^{\varepsilon} = \{\{p^{\epsilon}, W^{\epsilon}\}\}$$

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$\downarrow$ $\varepsilon \to 0$

$$\dot{W}_a^0 + \left\{\tau_a, W_a^0\right\} = \left[W_a^0, F_a\right]$$

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↓ $\varepsilon \to 0$

$$\dot{W}^0_a + \{\tau_a, W^0_a\} = [W^0_a, F_a]$$

$$W^0_a = \frac{1}{2}\begin{bmatrix} I+Q & U+iV \\ U-iV & I-Q\end{bmatrix} dq \wedge dp$$

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↓ unpolarized radiation

light transport equation

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↓ unpolarized radiation

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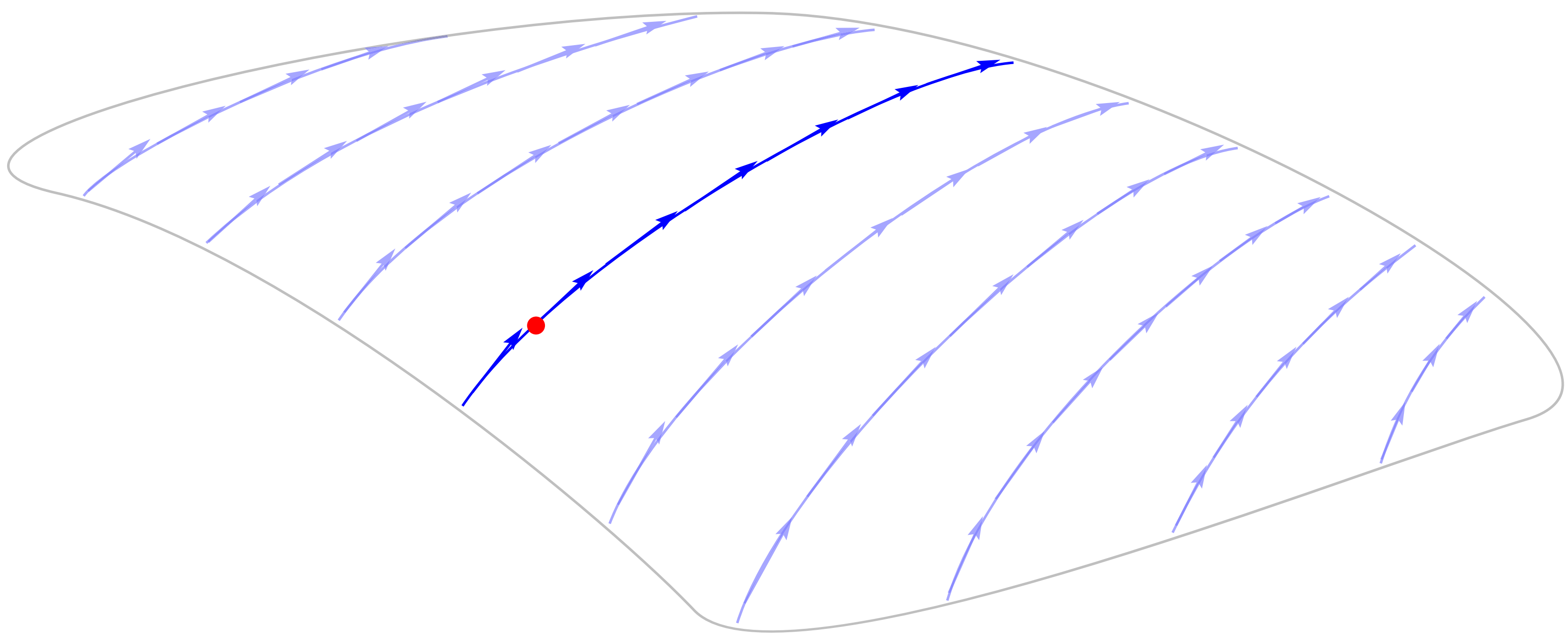
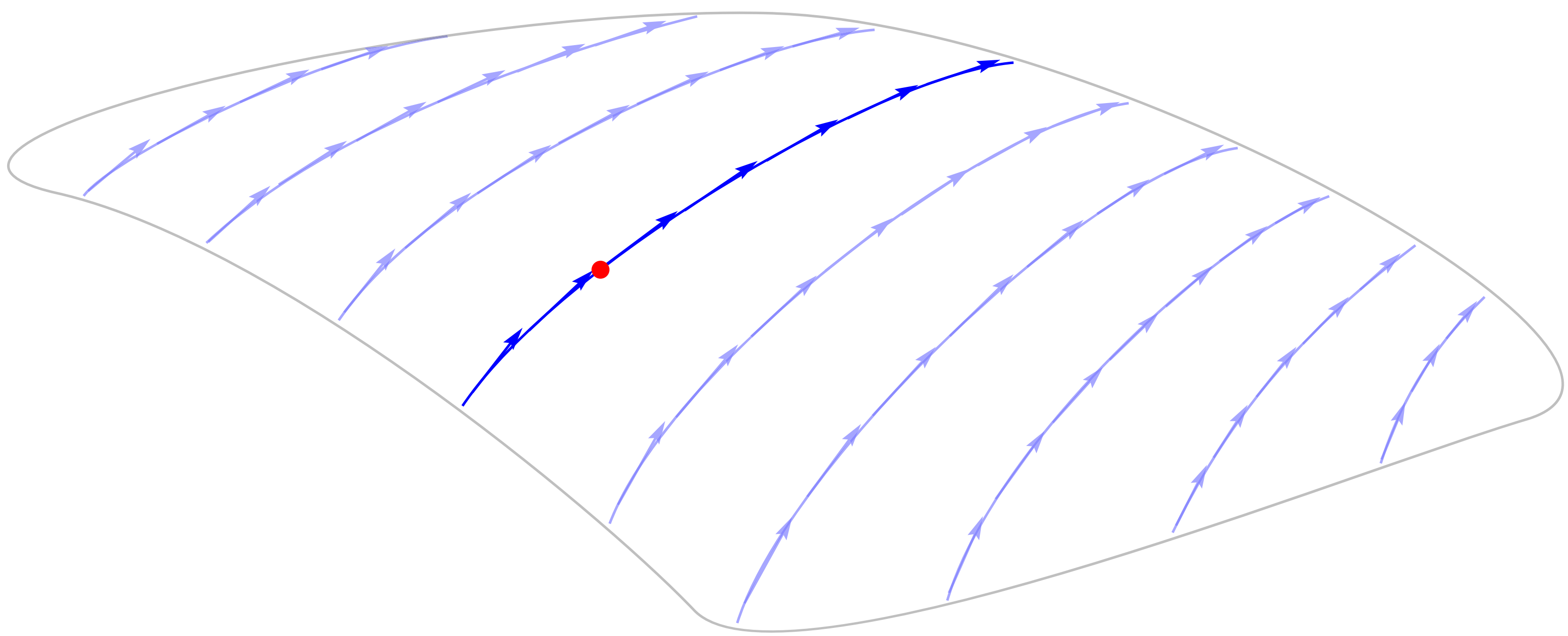
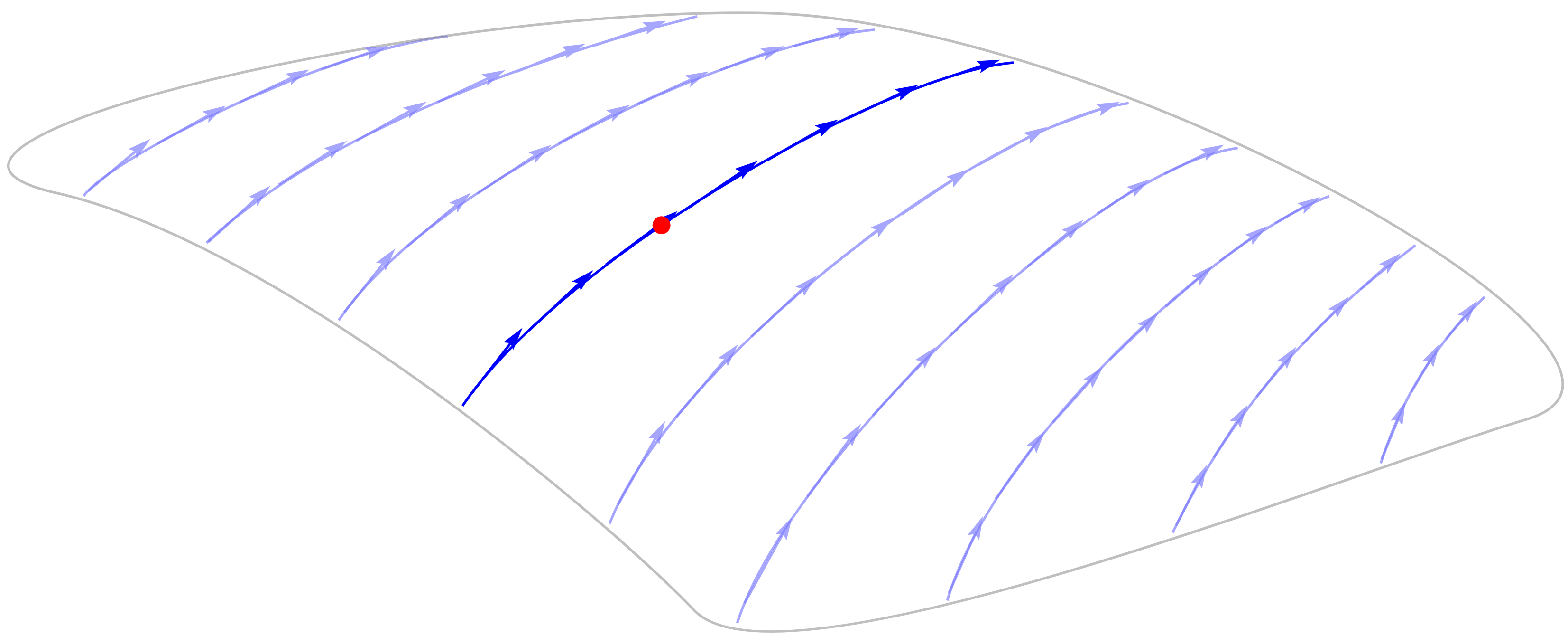
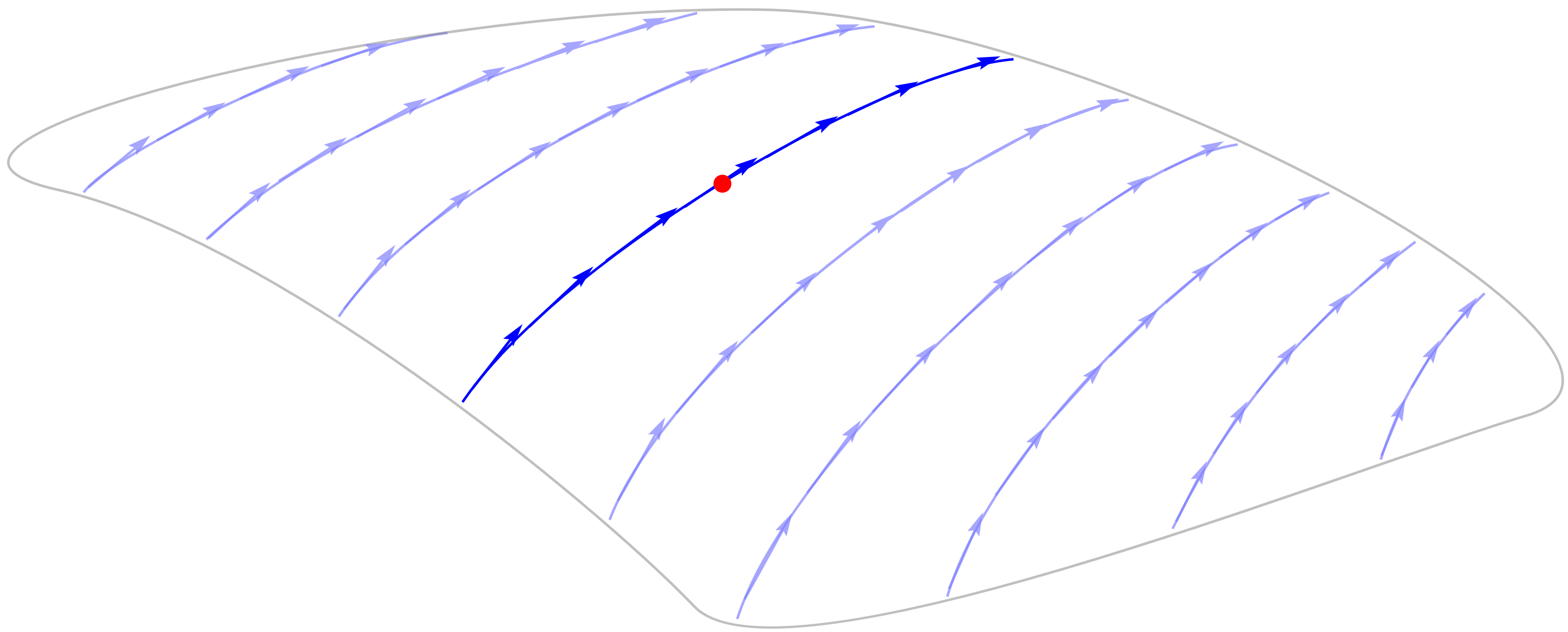
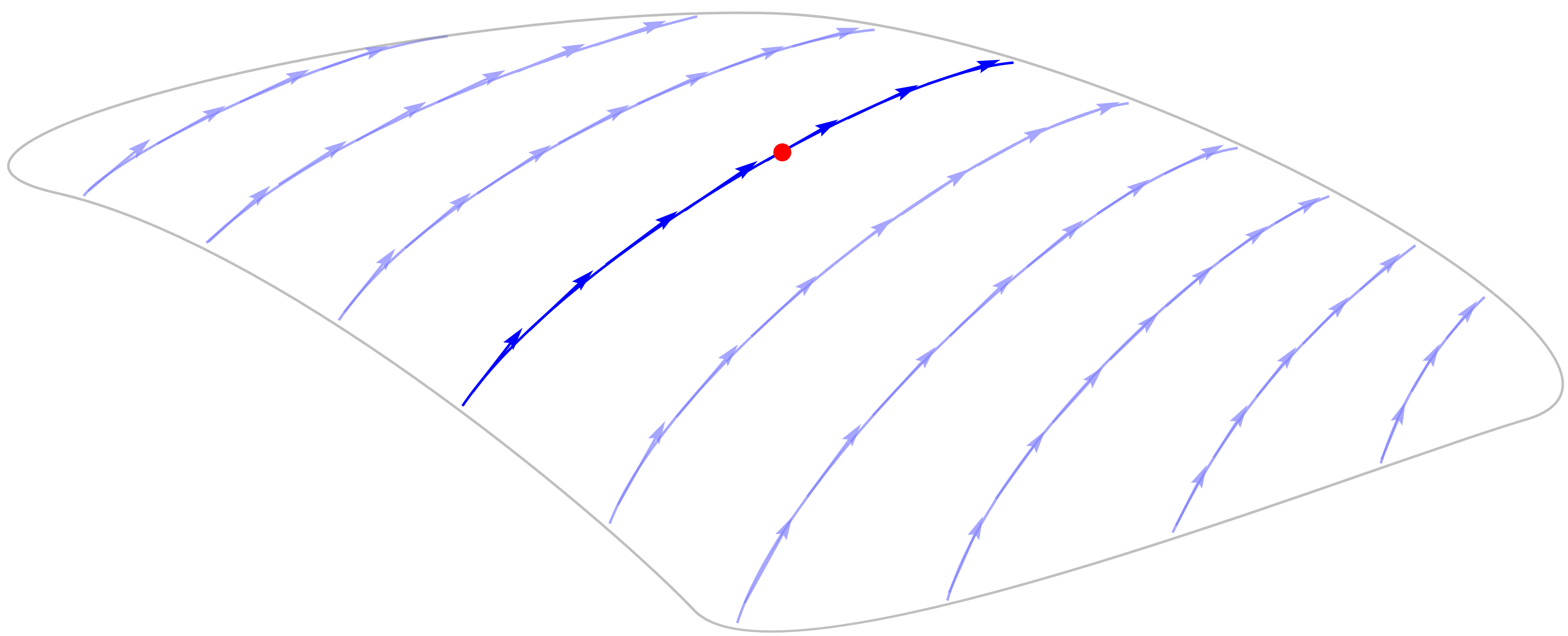
$$\ell = \mathcal{L}(q,p)\, dq \wedge dp$$

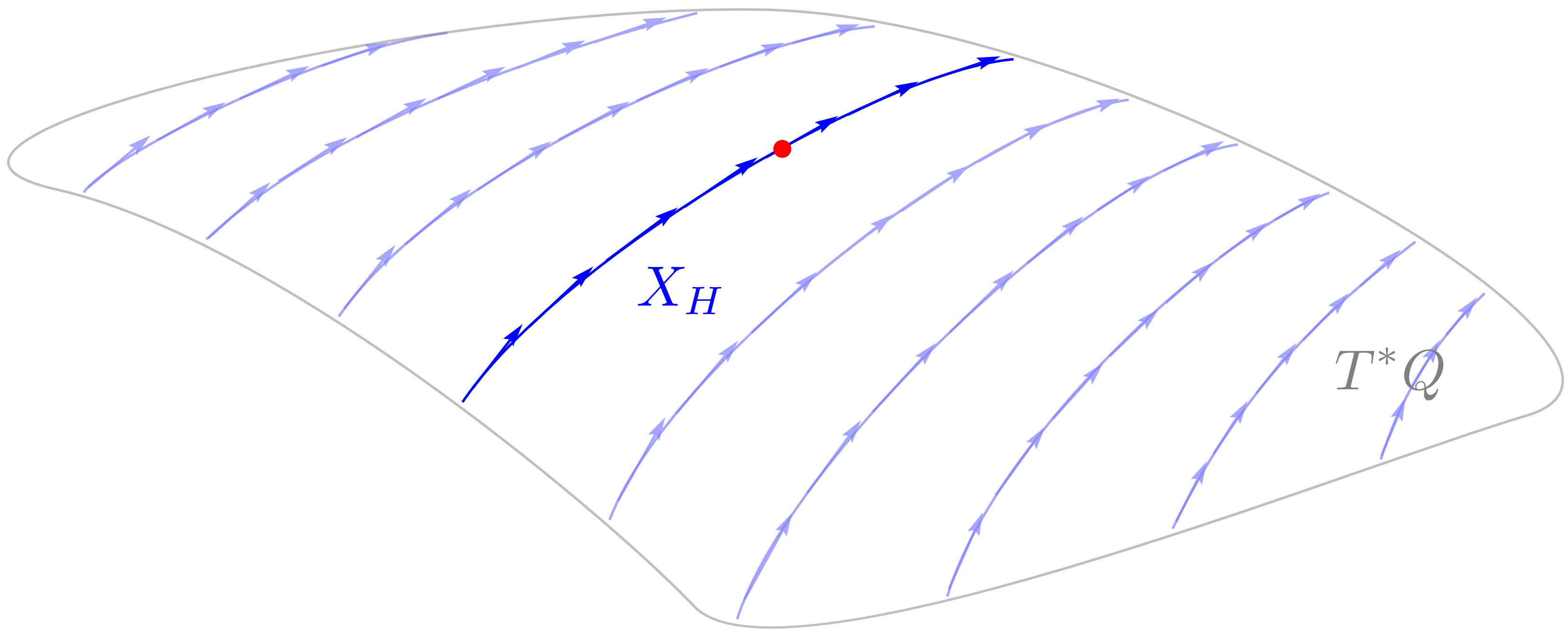


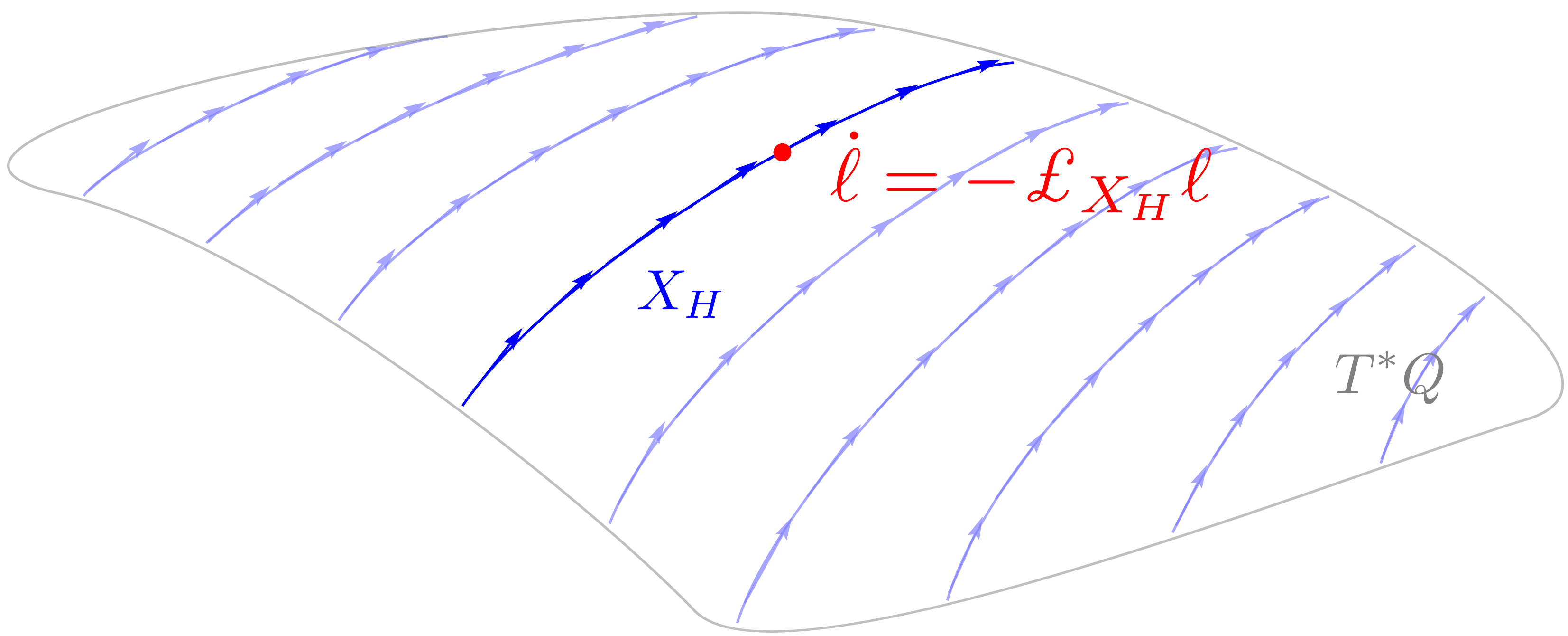


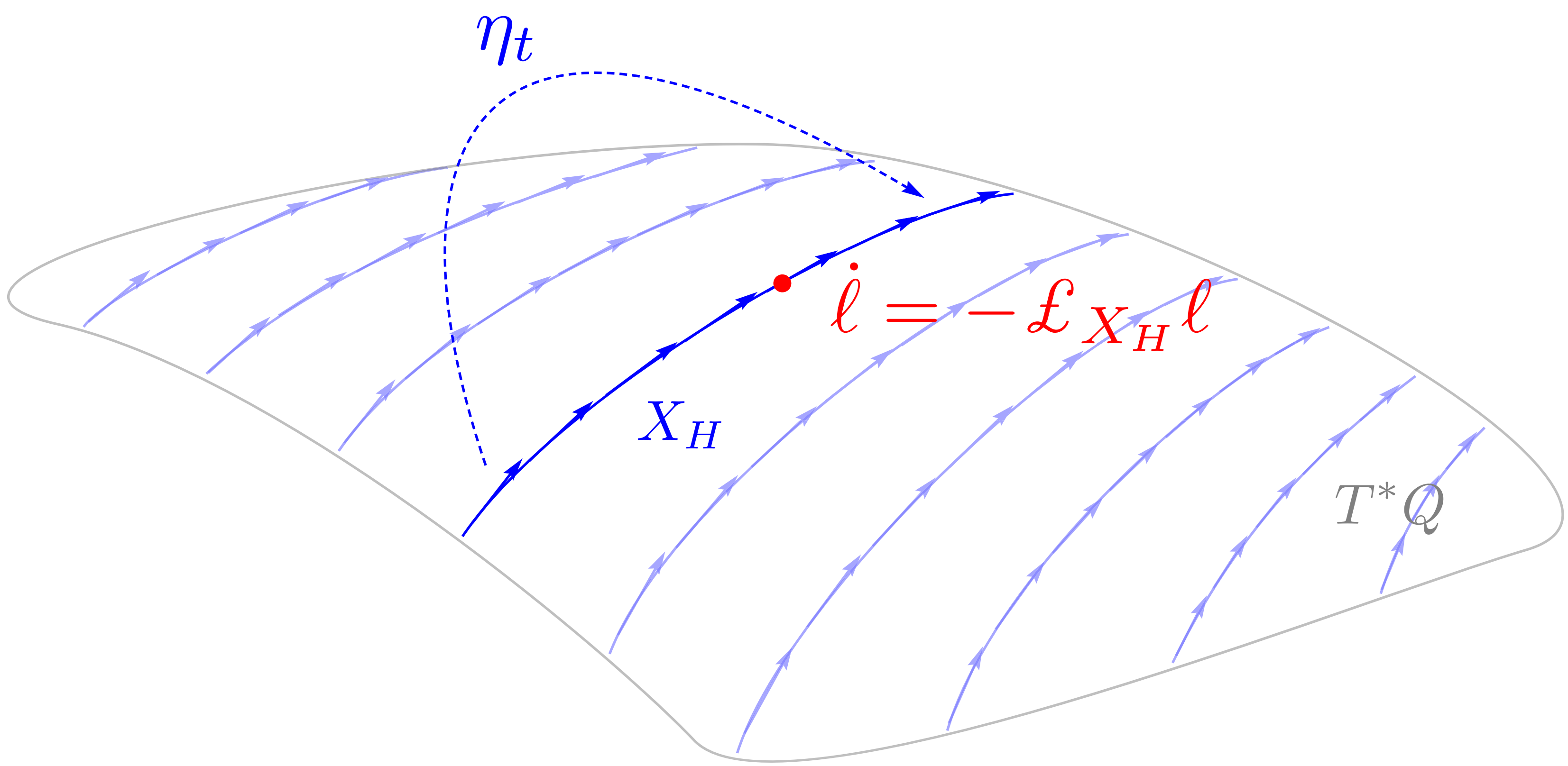


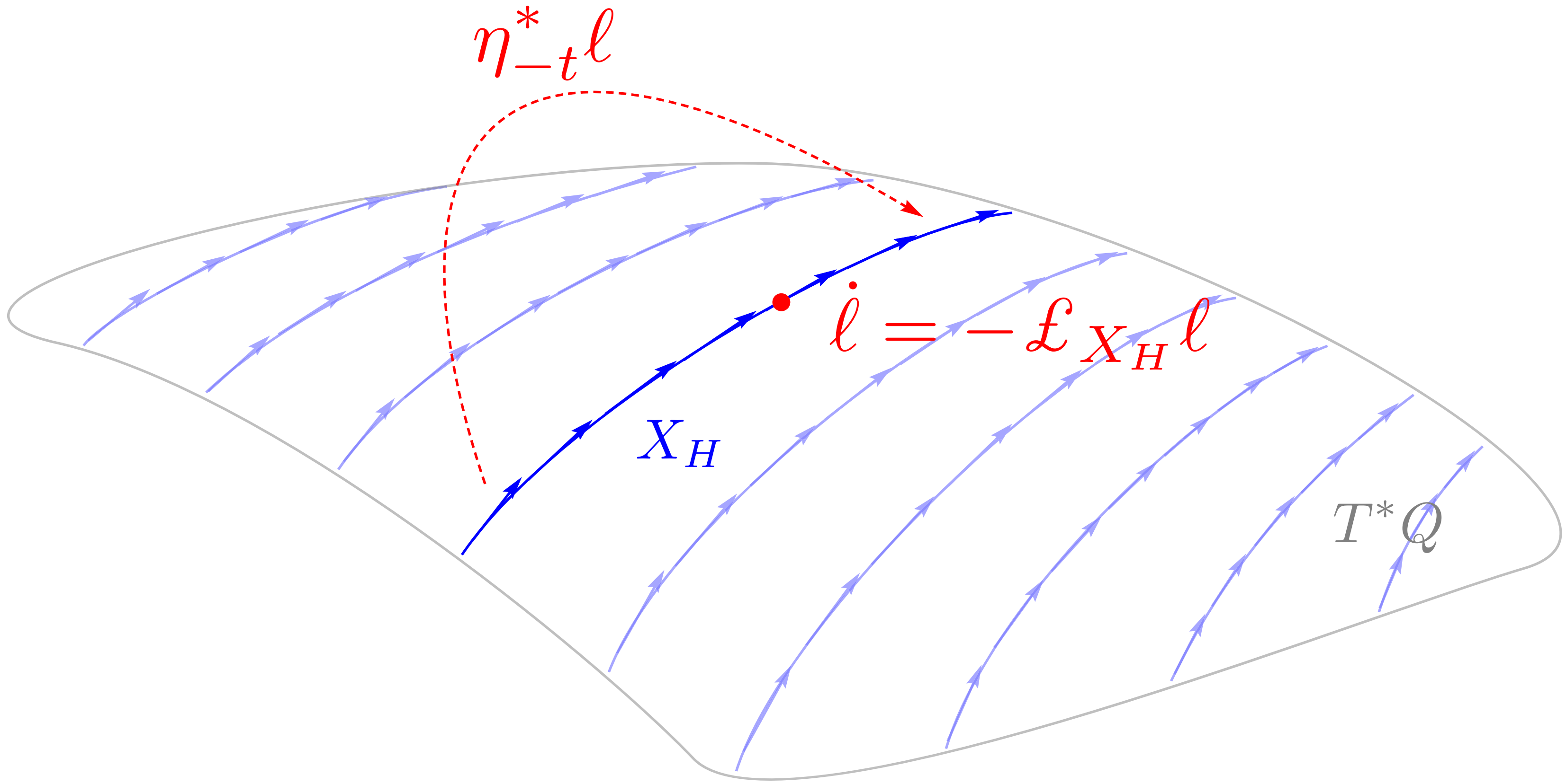





X_H
T^*Q


$\dot{\ell} = -\pounds_{X_H} \ell$
X_H
T^*Q


η_t
$\dot{\ell} = -\pounds_{X_H}\ell$
X_H
T^*Q


$\eta^*_{-t}\ell$
$\dot{\ell} = -\pounds_{X_H}\ell$
X_H
T^*Q

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↓ $\varepsilon \to 0$

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light transport equation

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← Legendre transform

Fermat's principle

$$\hat{L} = n^2(q)$$

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conservation of frequency

↓ unpolarized radiation

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← Legendre transform

Fermat's principle

$$\hat{L} = n^2(q)$$

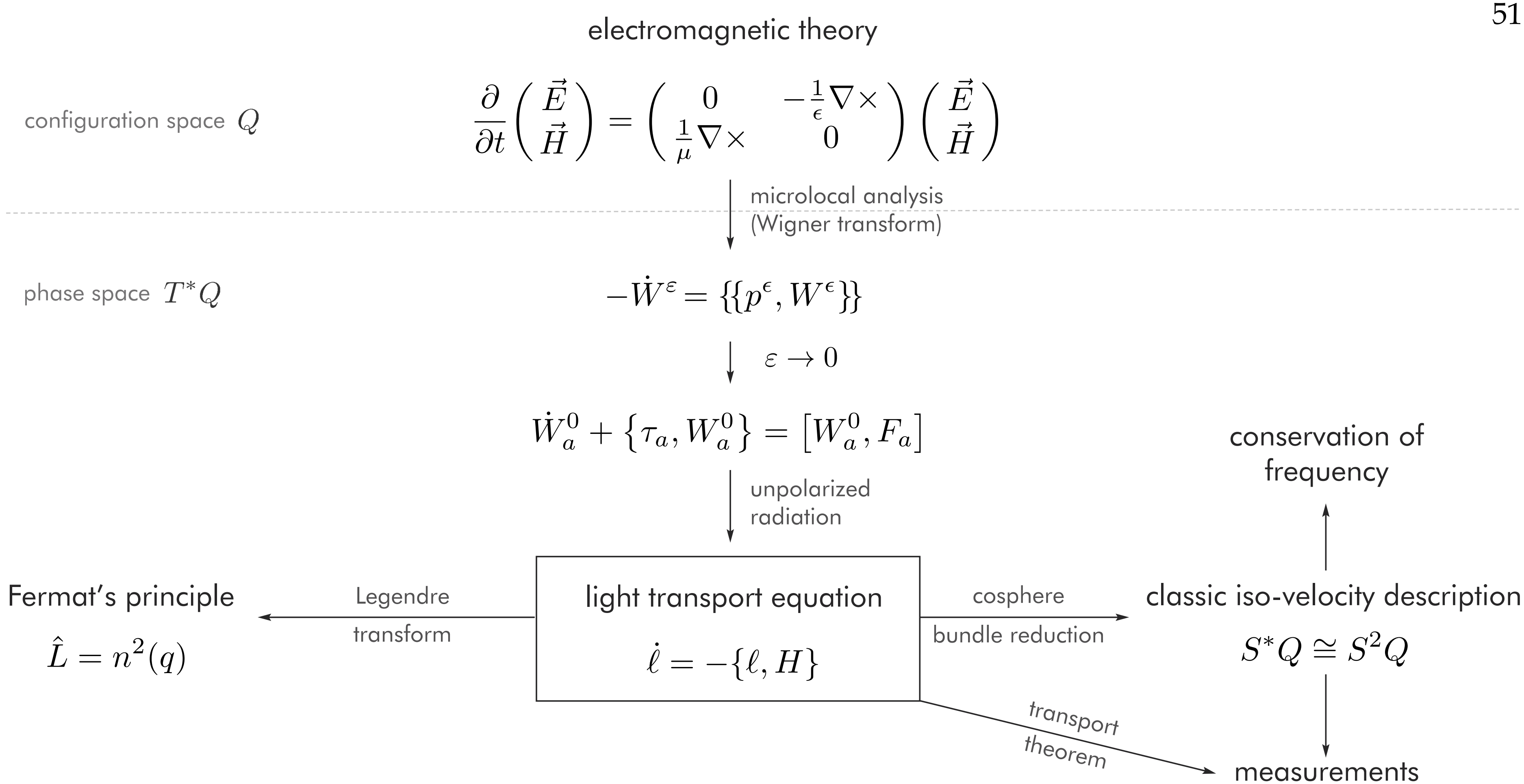
→ cosphere bundle reduction

classic iso-velocity description

$$S^*Q \cong S^2Q$$

↑

conservation of frequency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light transport equation
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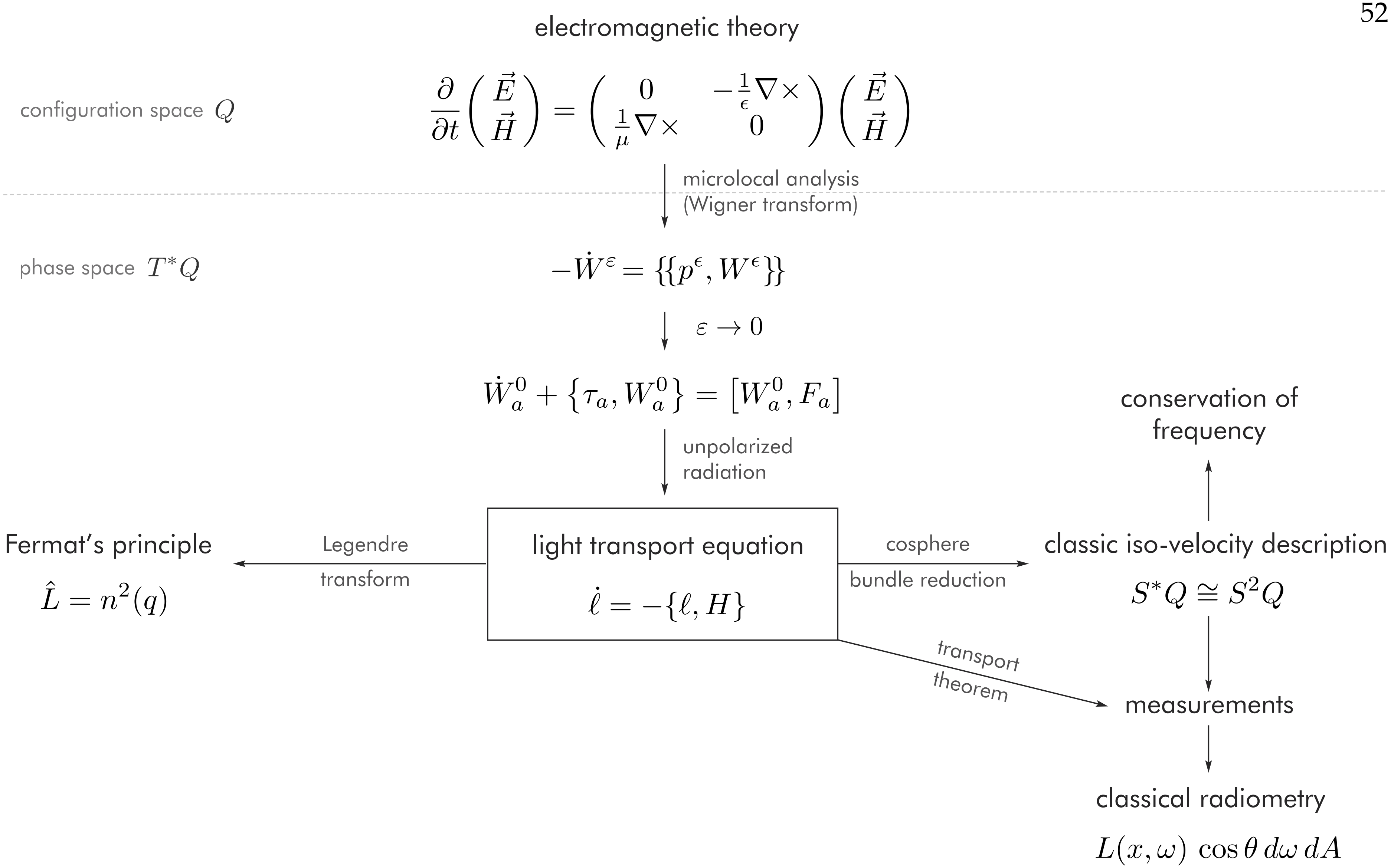
cosphere
bundle reduction
classic iso-velocity description
$S^*Q \cong S^2Q$
transport
theorem
measurements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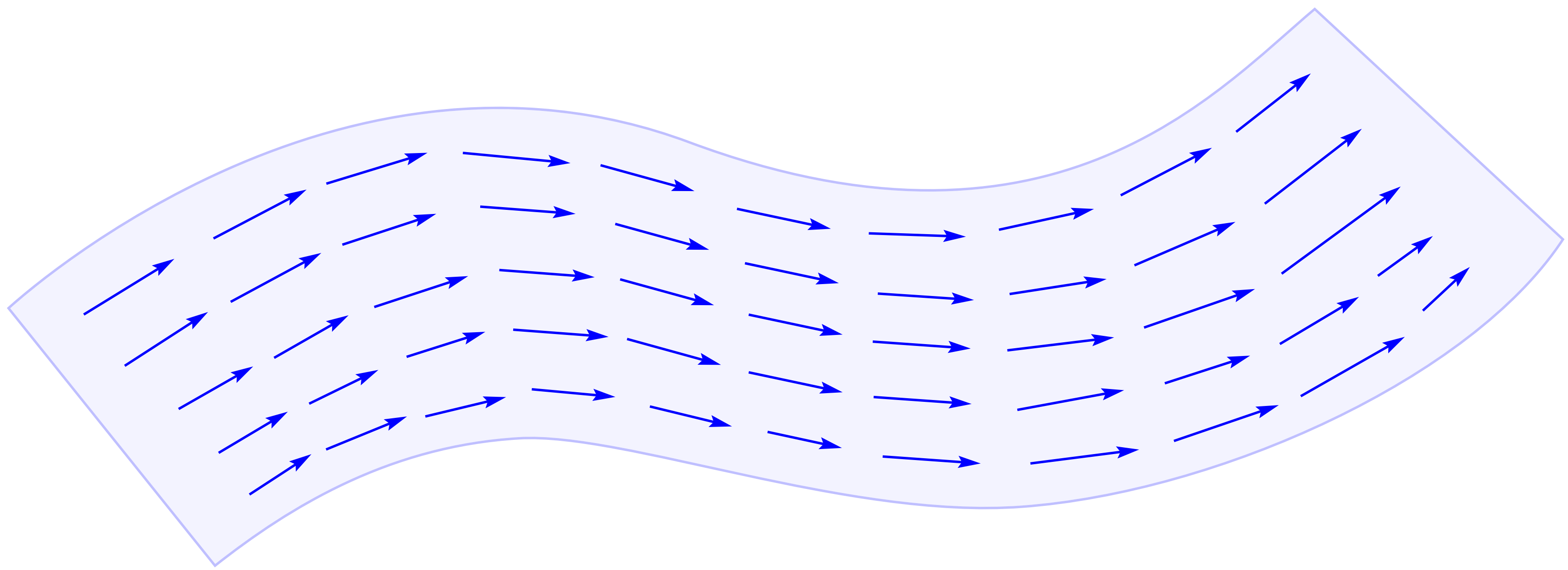
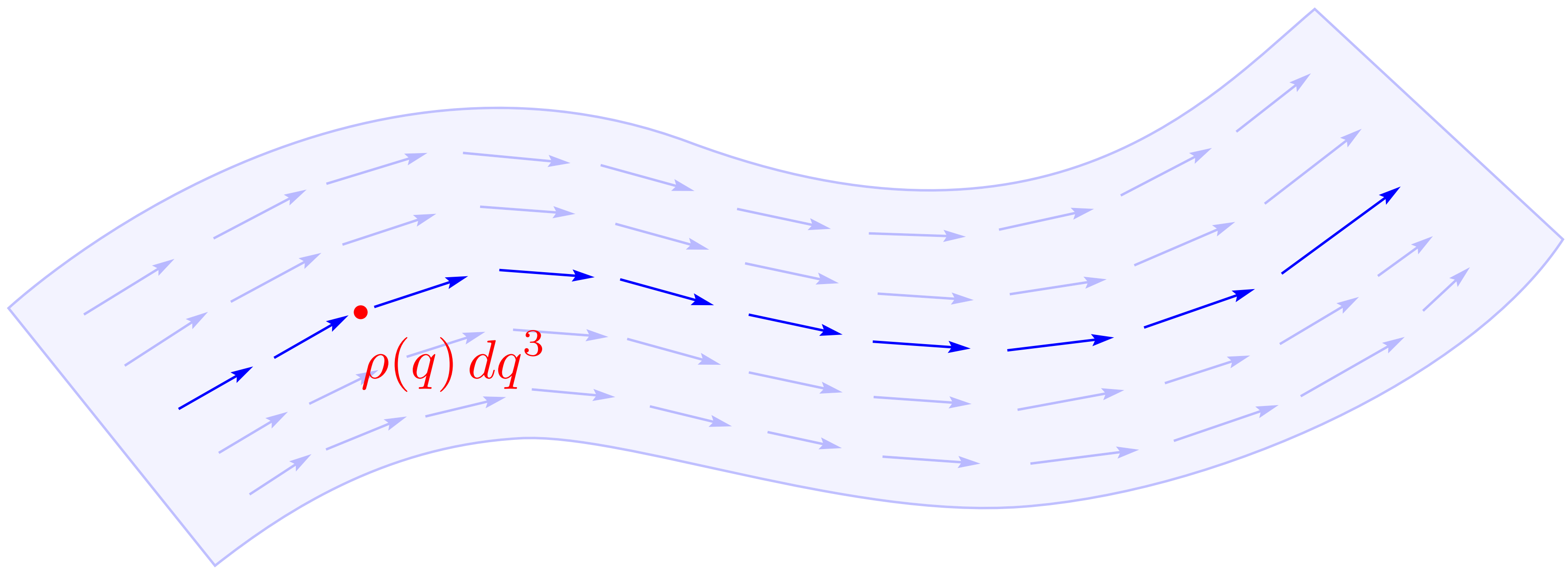
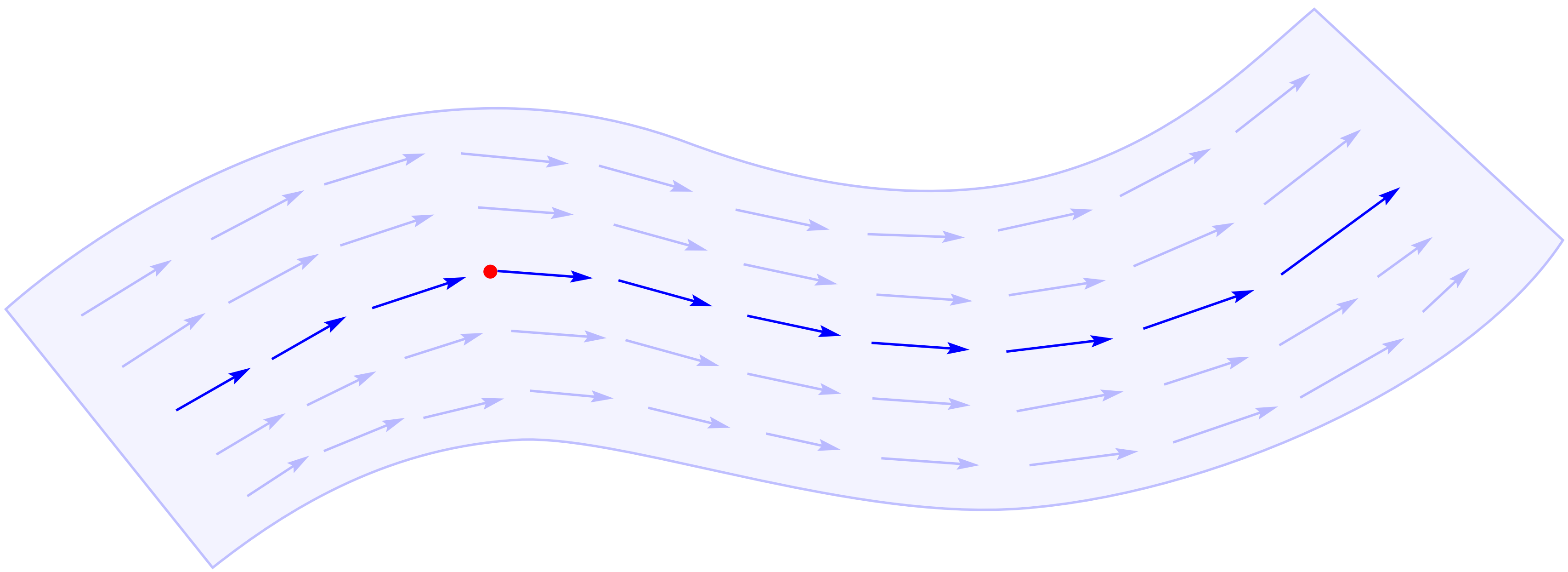
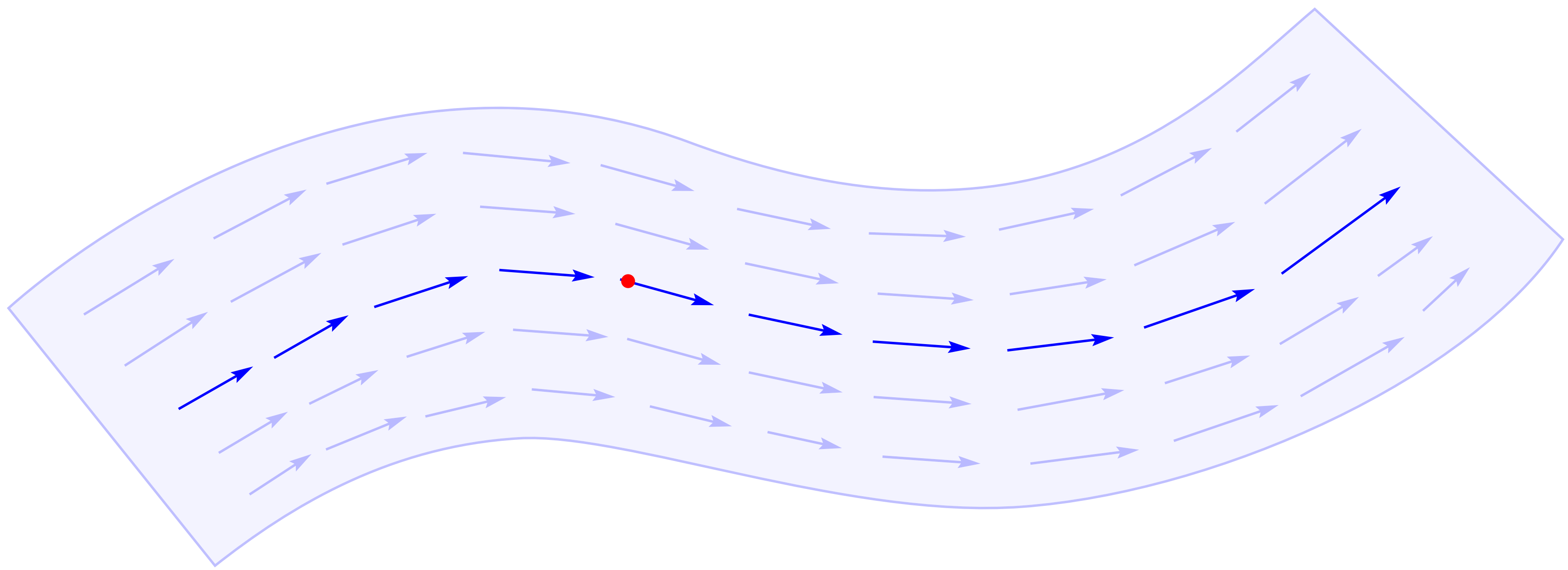
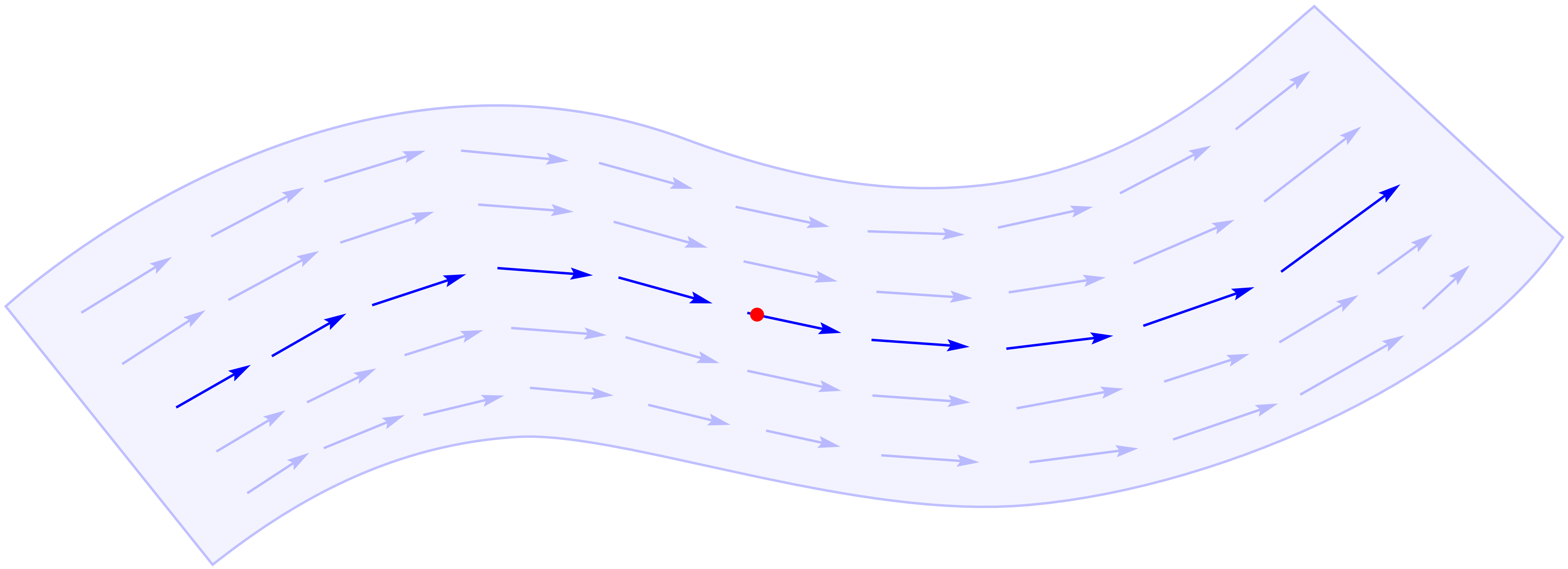
classical radiometry
$L(x,\omega)\ \cos\theta\, d\omega\, dA$

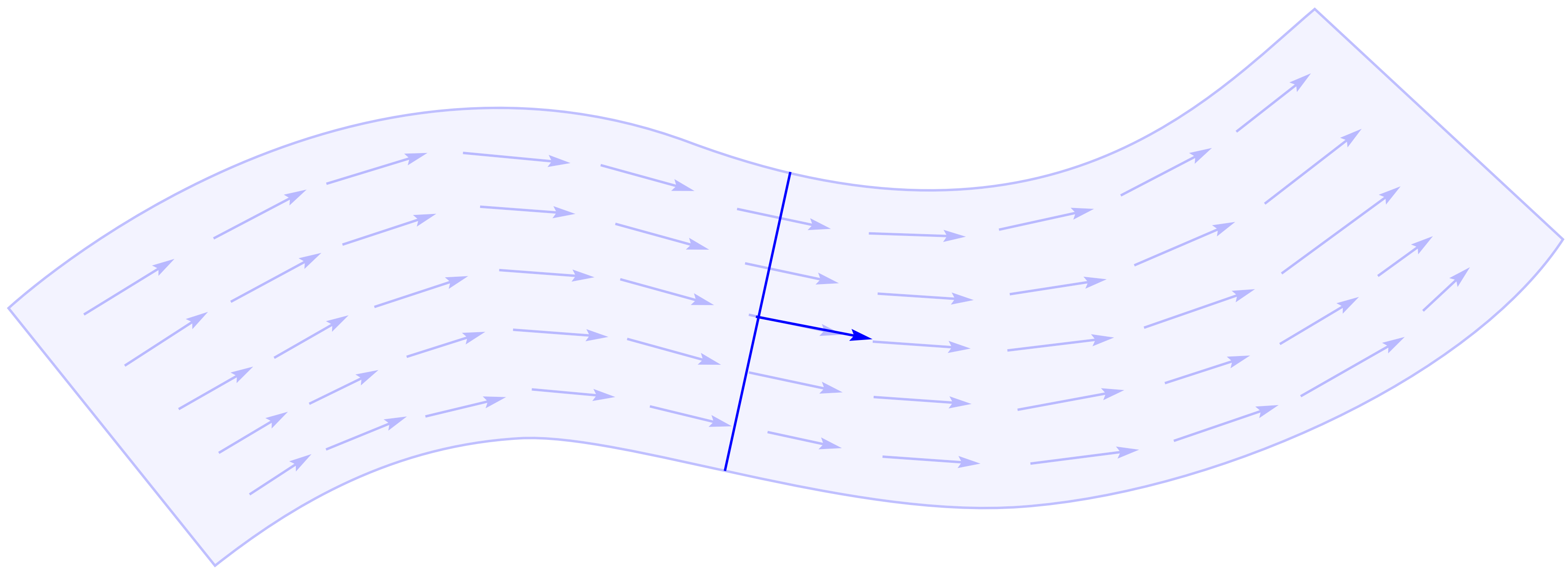


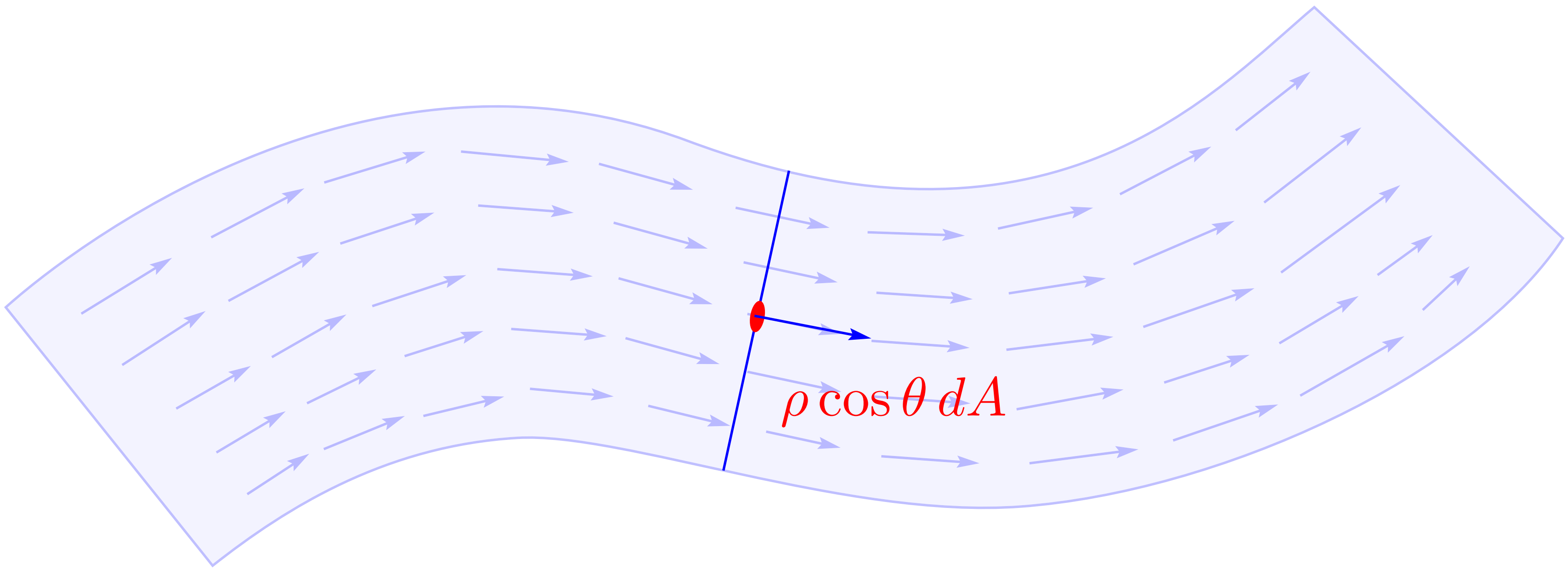

$\rho(q)\,dq^3$

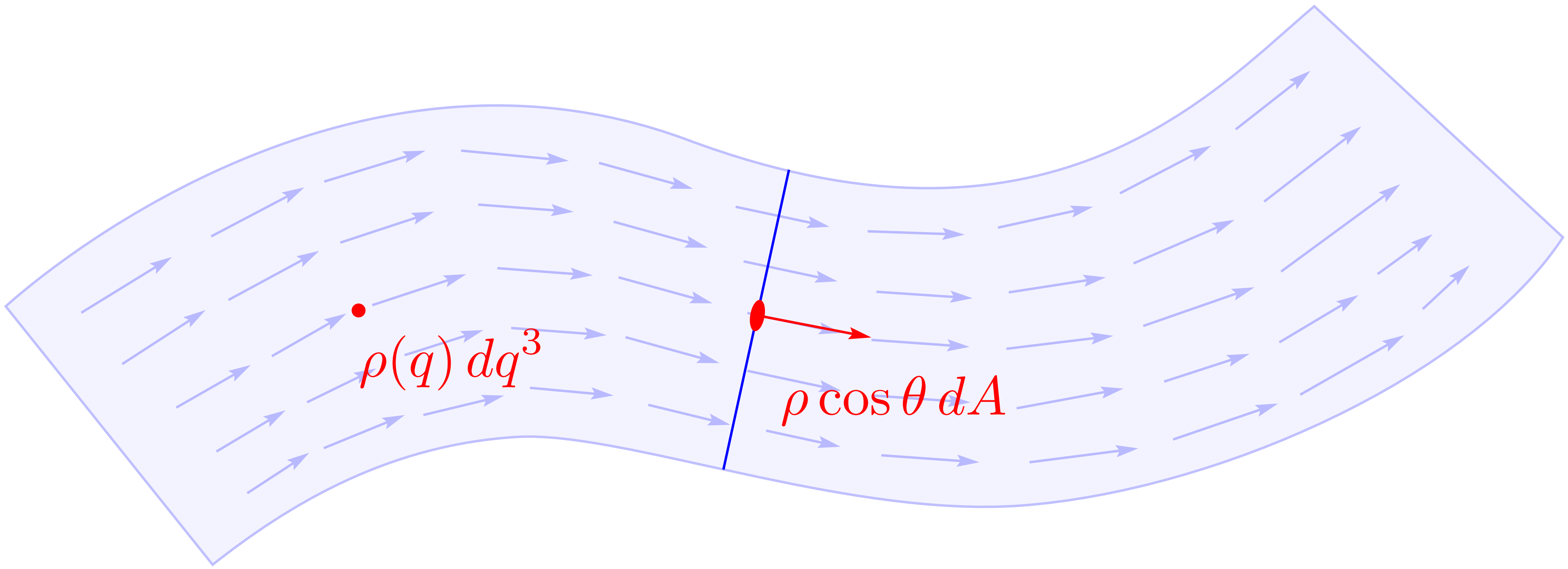









$\rho \cos\theta \, dA$



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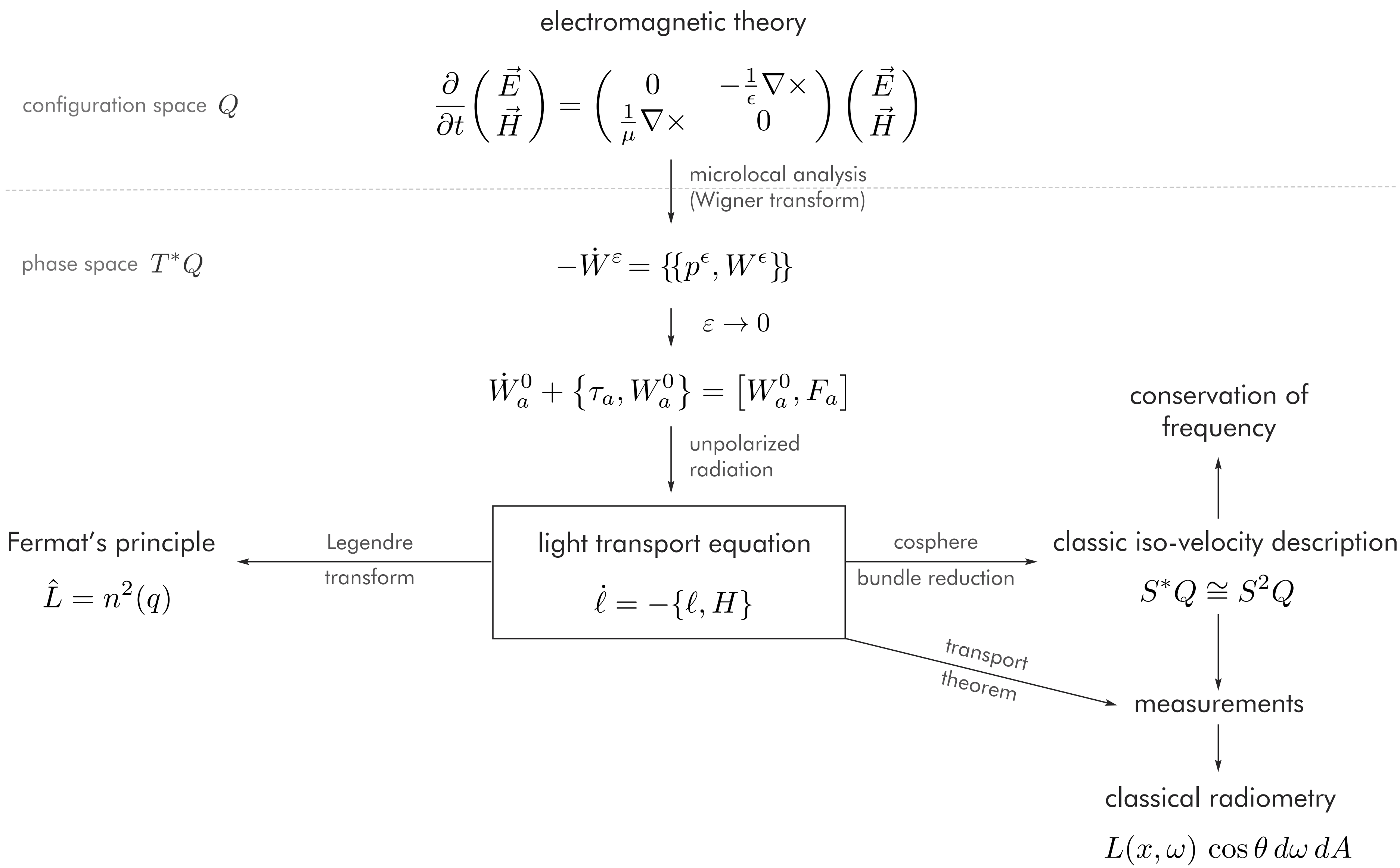
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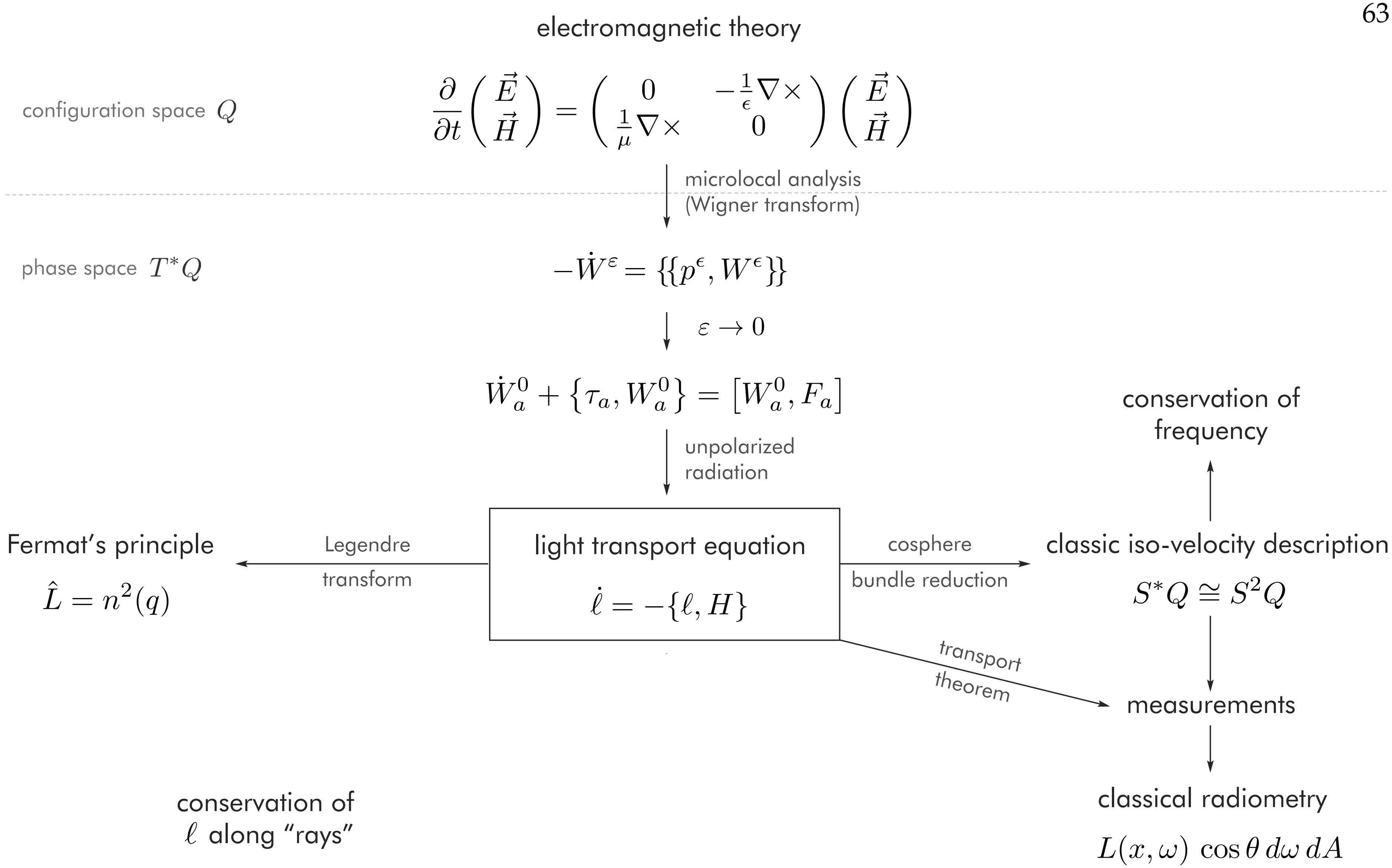
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$\dot{W}^0_a + \{\tau_a, W^0_a\} = [W^0_a, F_a]$
unpolarized
radiation
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$\dot{\ell} = -\{\ell, H\}$
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$\hat{L} = n^2(q)$
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conservation of
frequency
transport
theorem
measurements
classical radiometry
$L(x,\omega)\ \cos\theta\, d\omega\, dA$
conservation of
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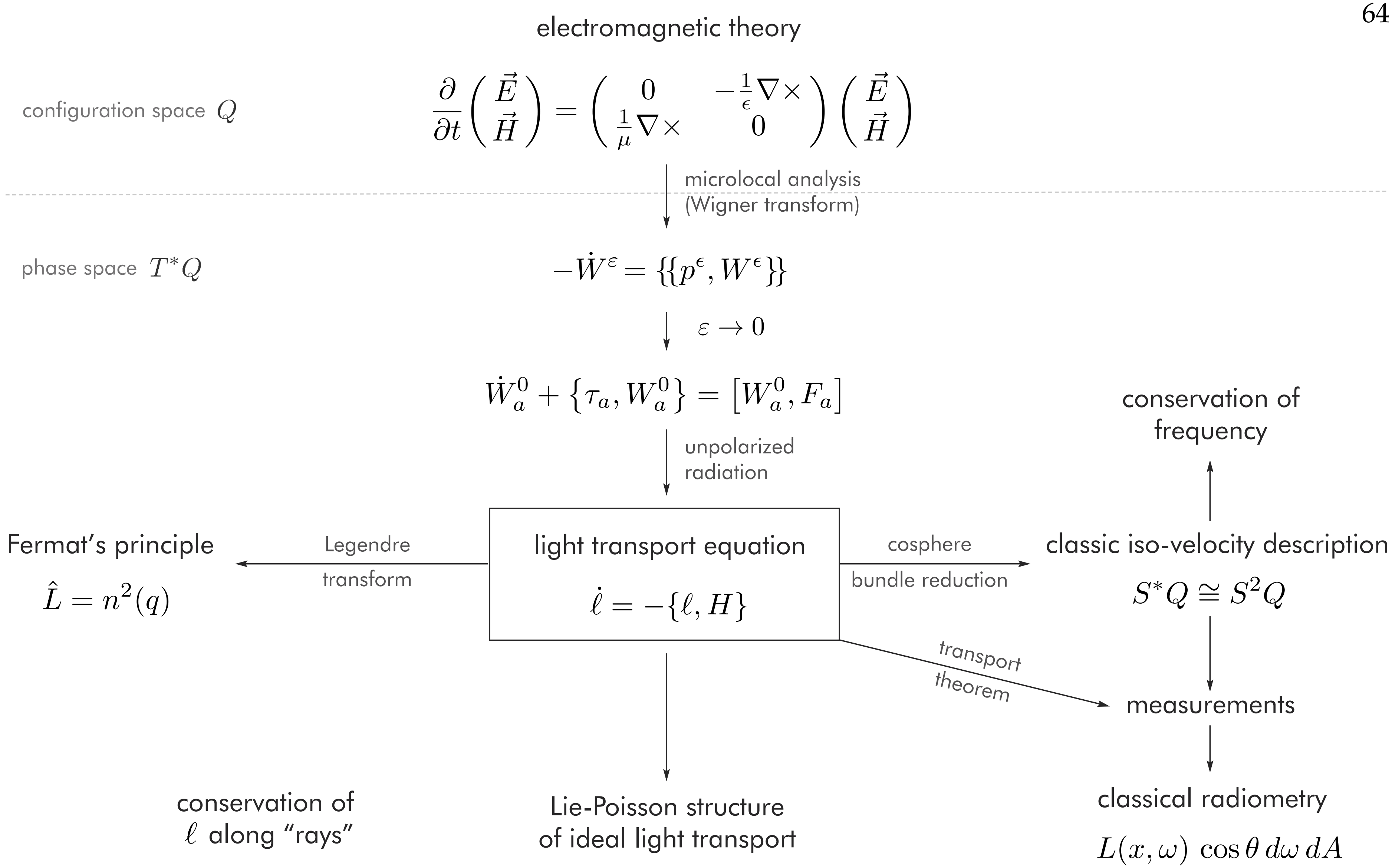
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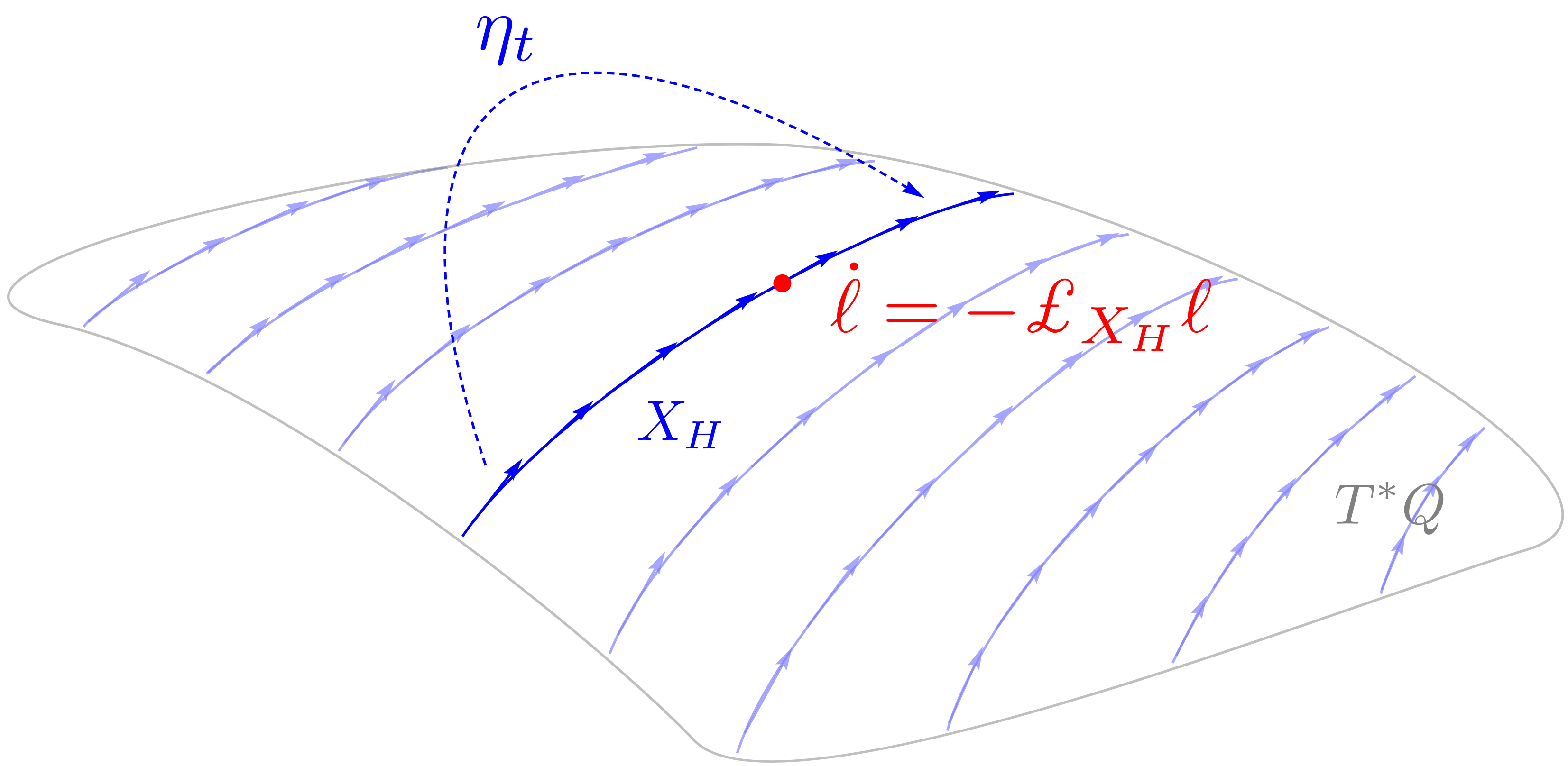
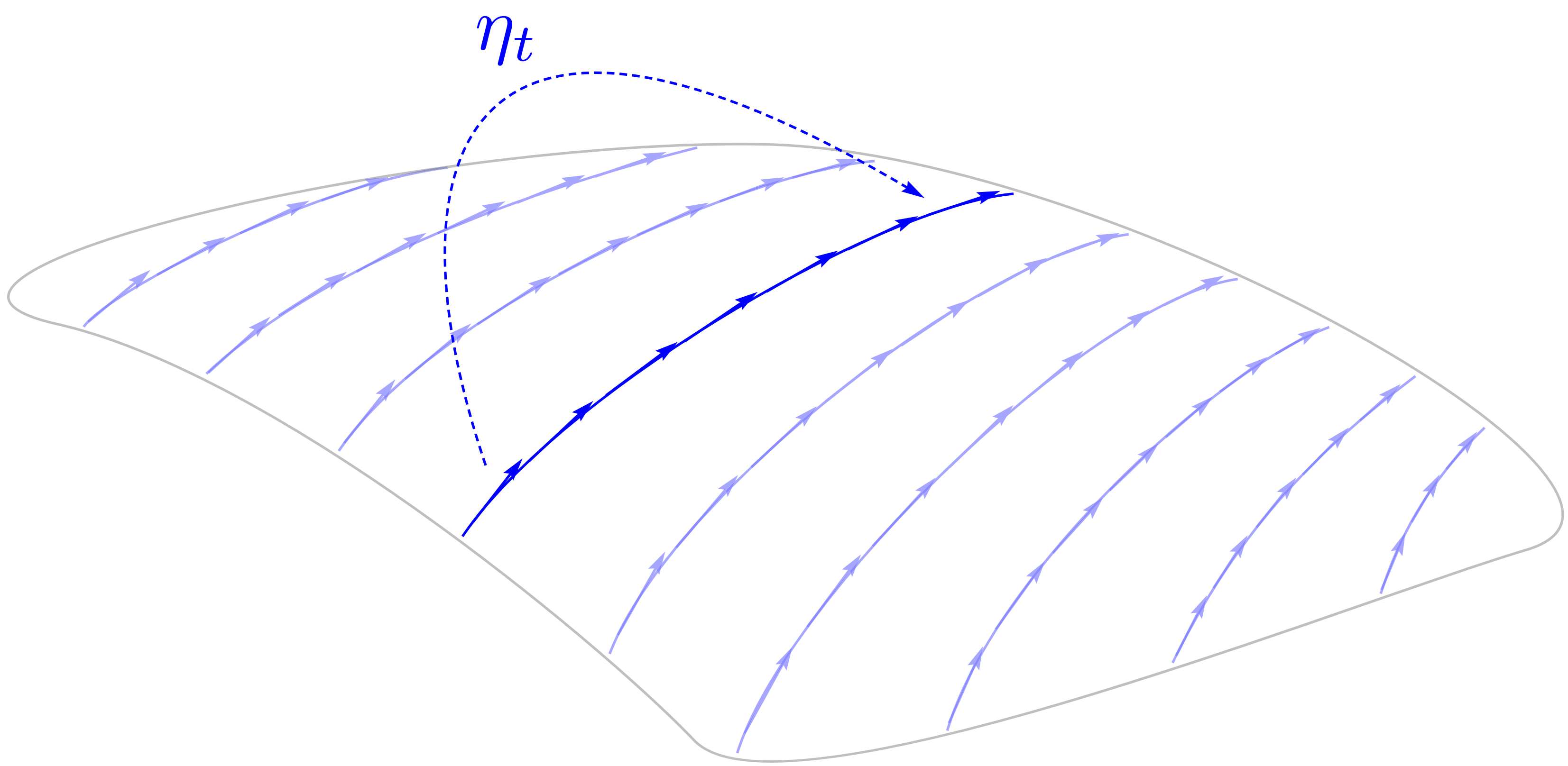
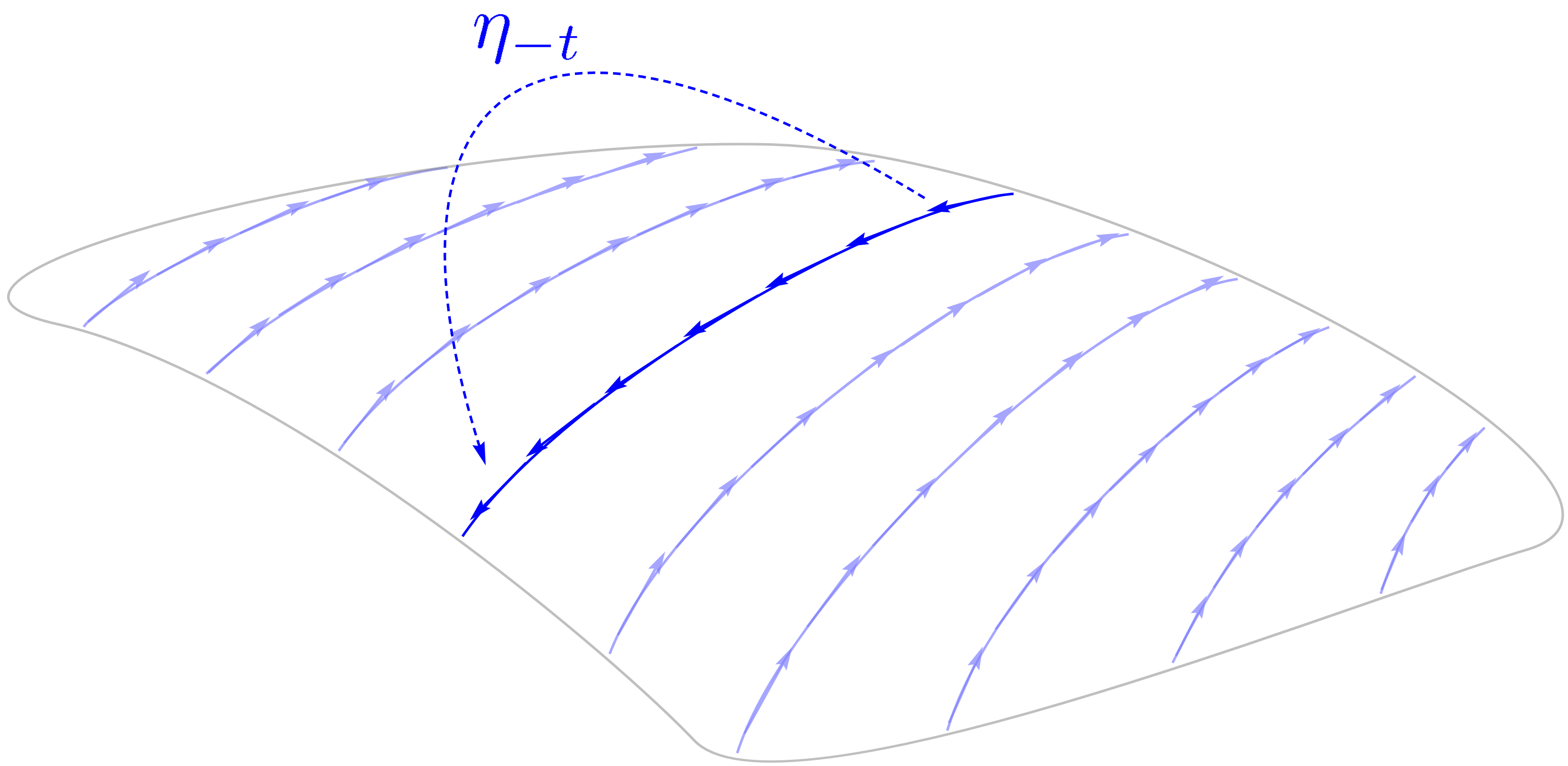
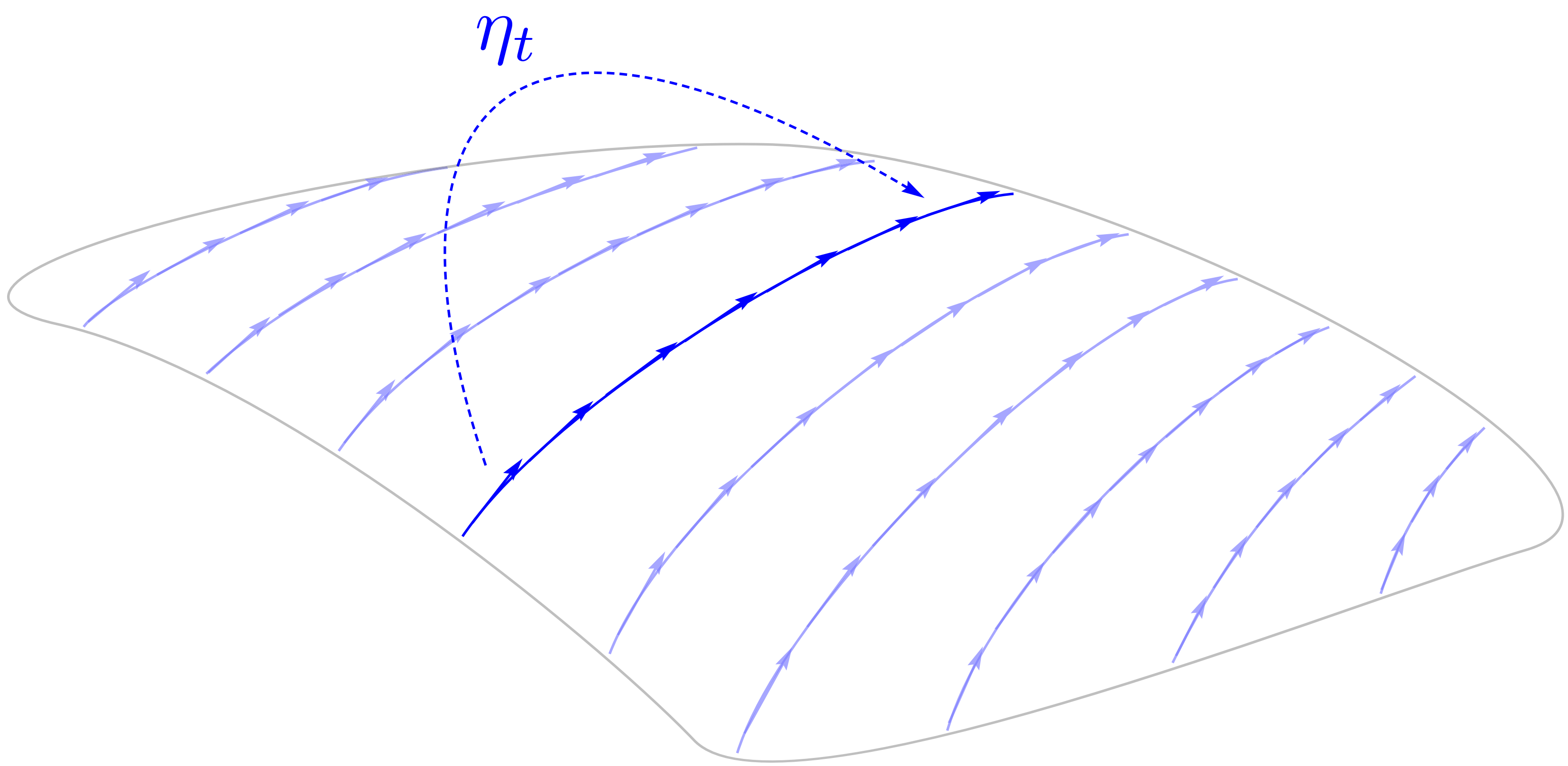
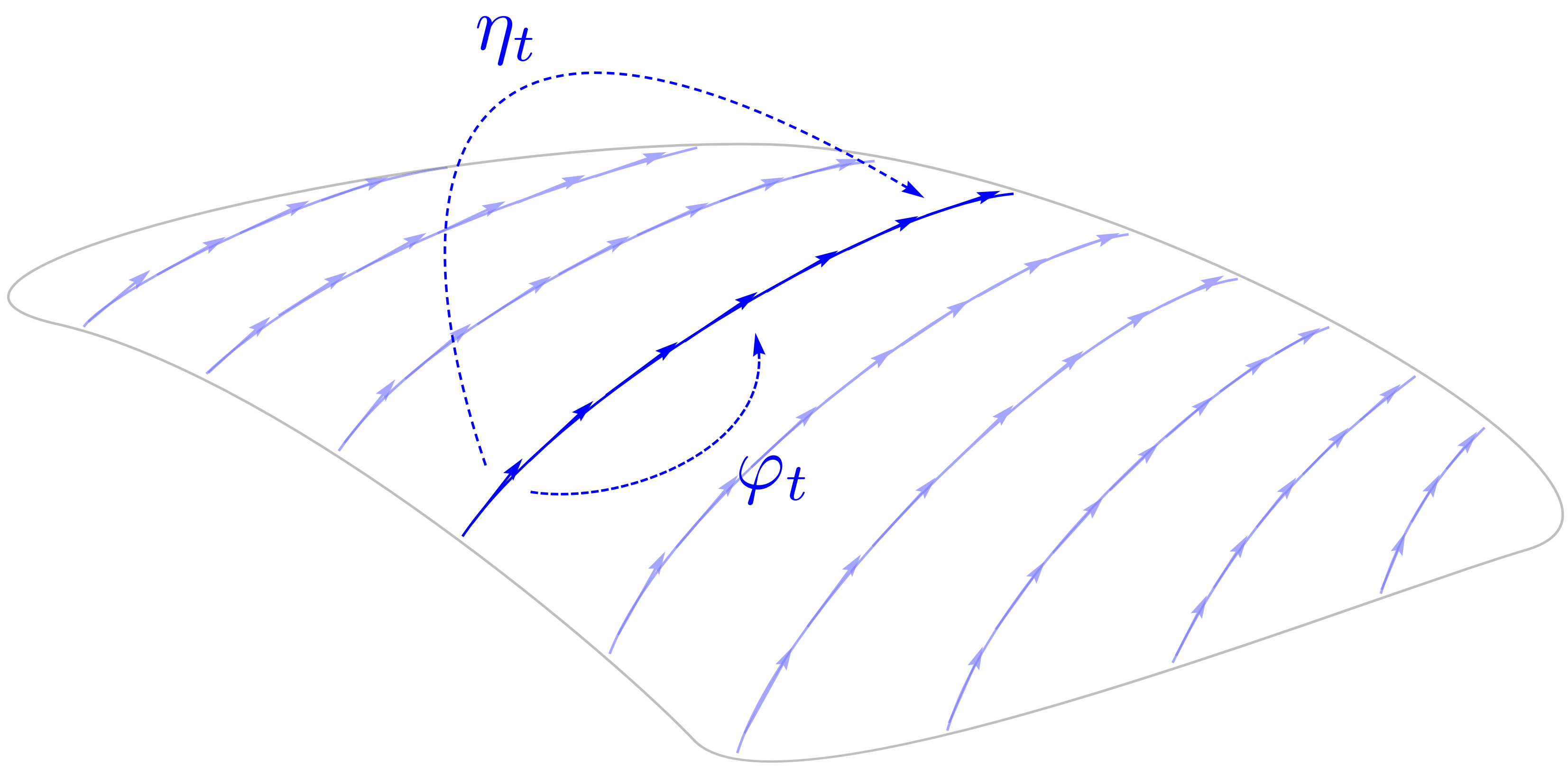


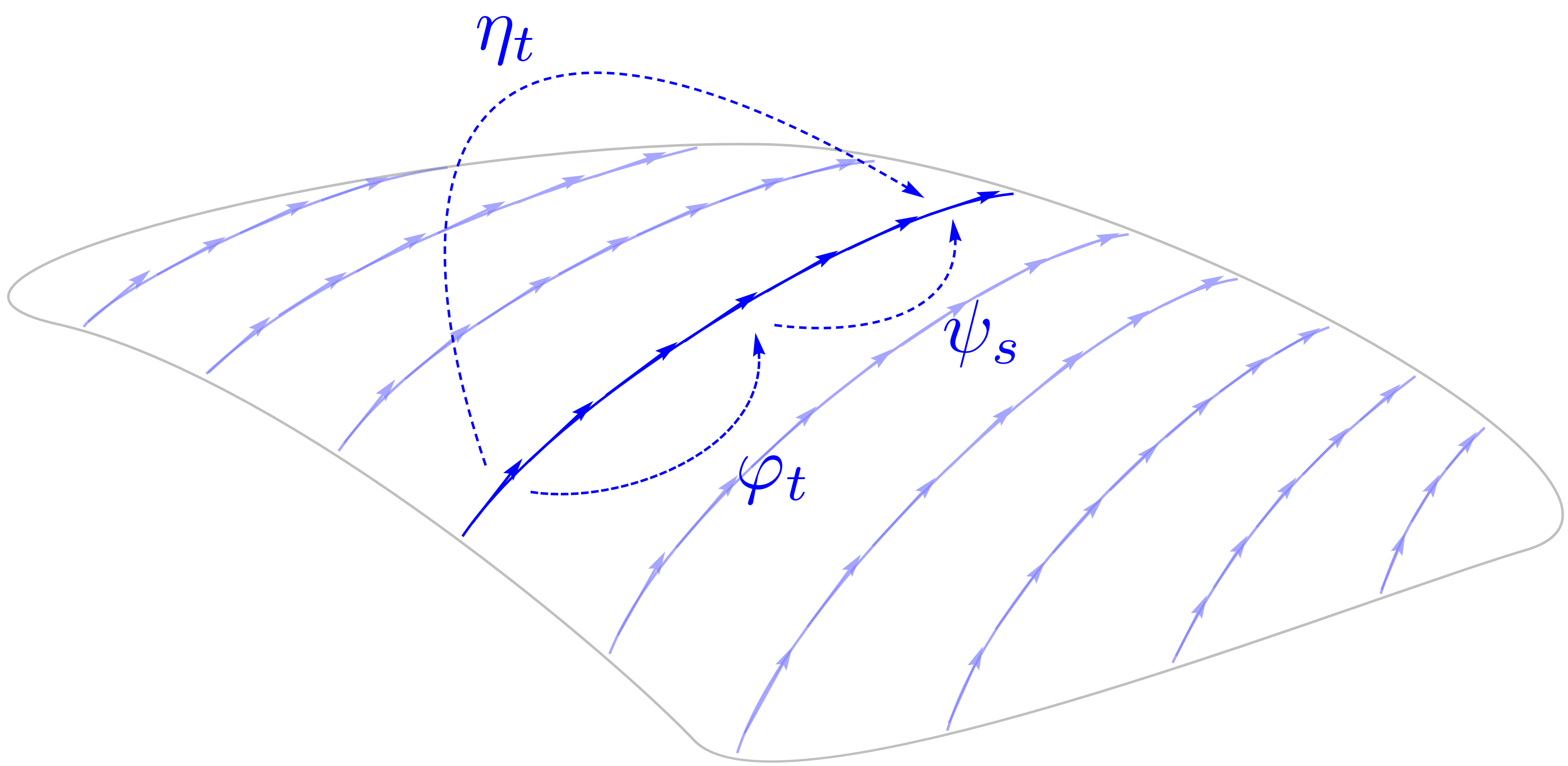
η_t
$\dot{\ell} = -\pounds_{X_H}\ell$
X_H
T^*Q


η_t

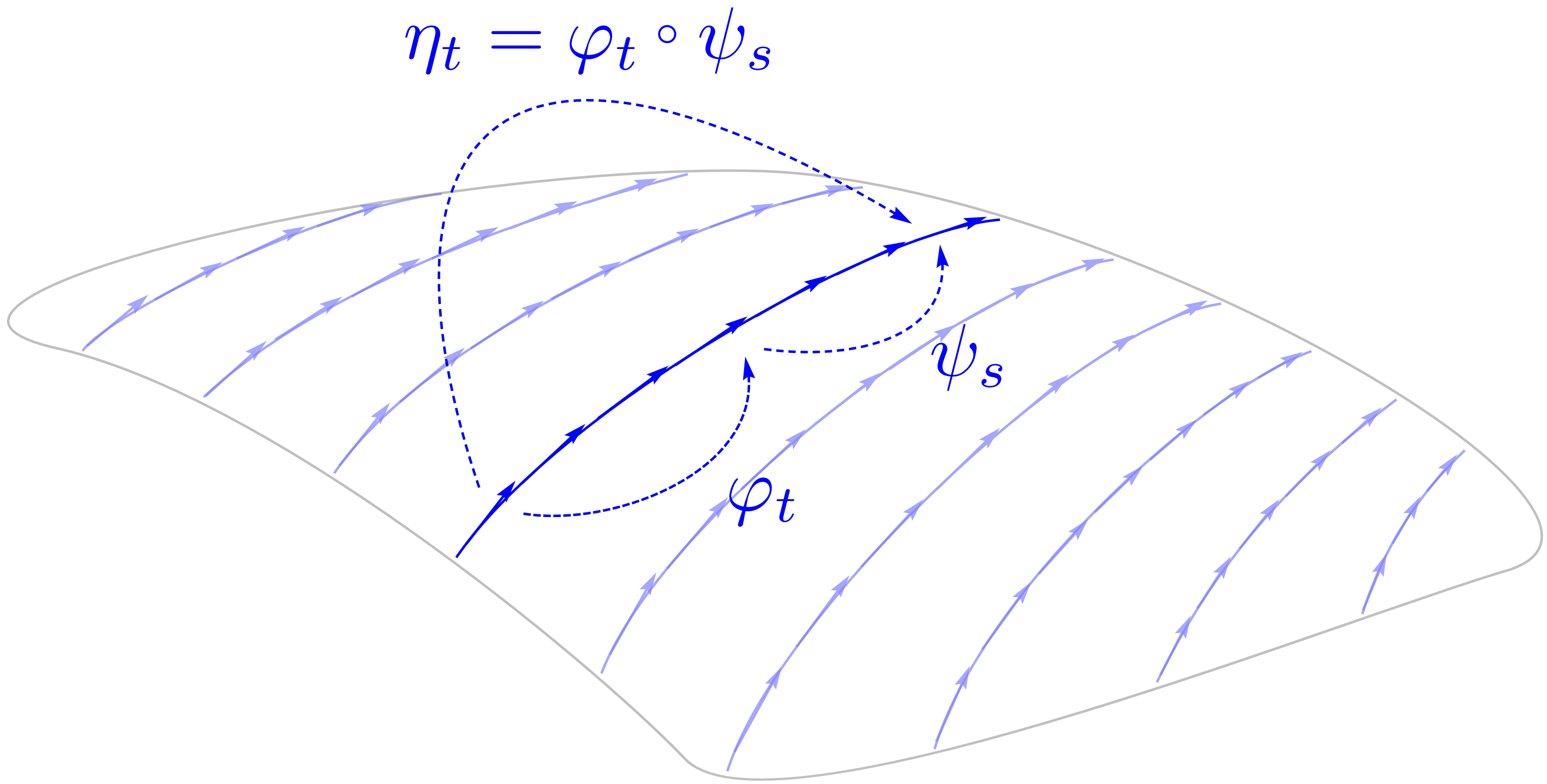

η_{-t}

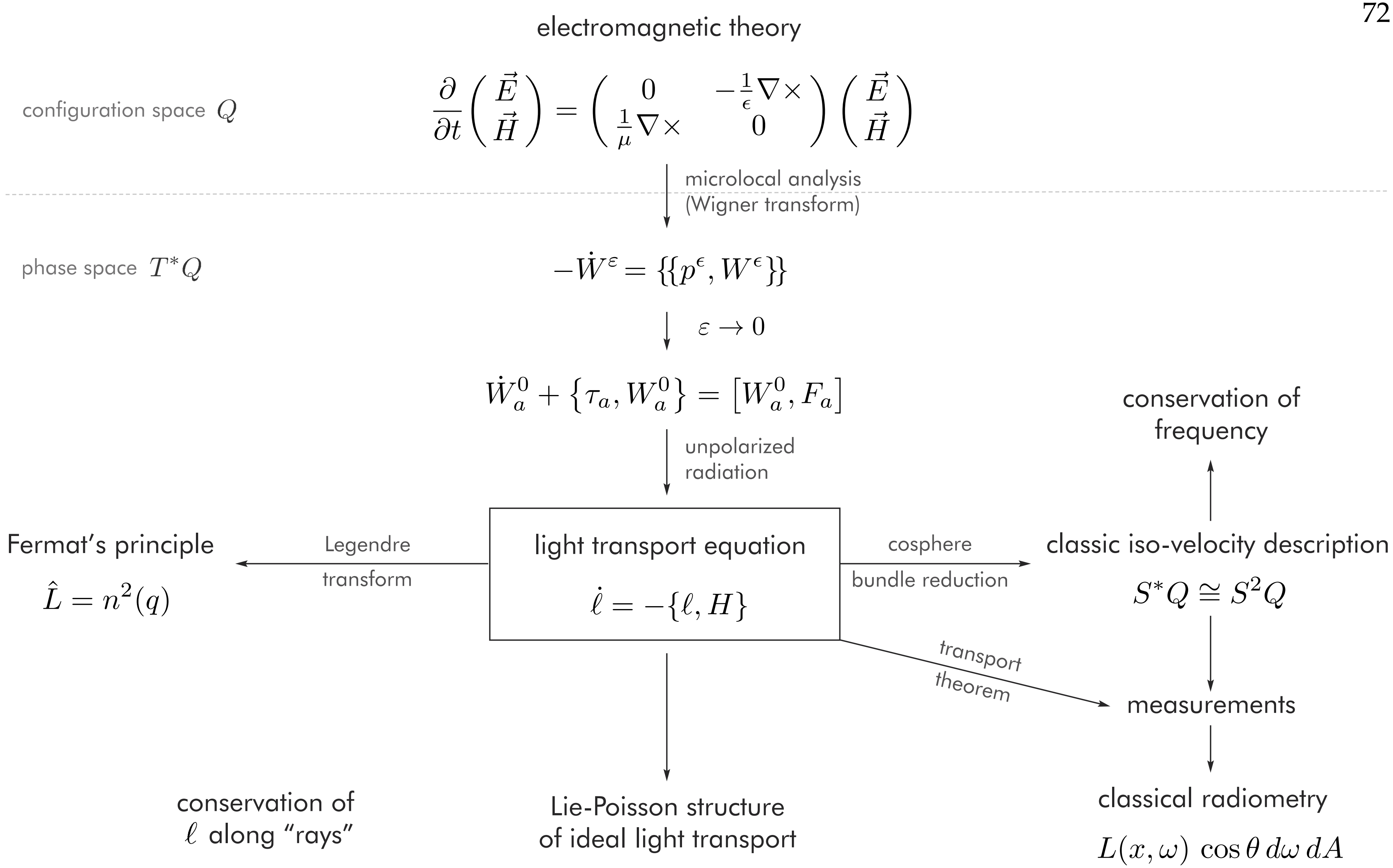

η_t


η_t
φ_t



η_t
ψ_s
φ_t


$\eta_t = \varphi_t \circ \psi_s$
ψ_s
φ_t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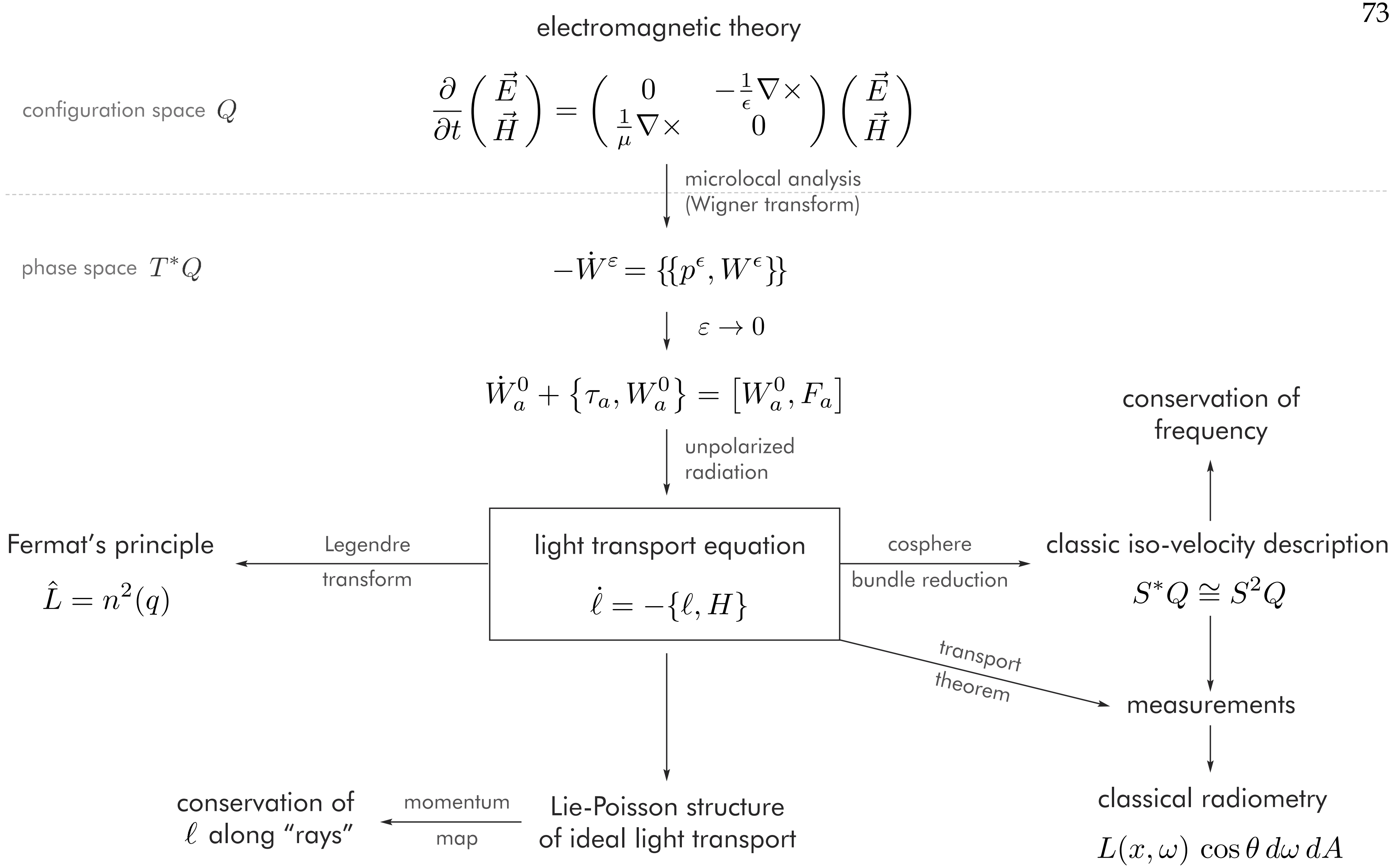
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	ideal fluid dynamics	ideal light transport
Lie group	$\mathrm{Diff}_{\mu}(Q)$	$\mathrm{Diff}_{\mathrm{can}}(T^*Q)$
dual Lie algebra representation	vorticity	light energy density
coadjoint action	$\dot{\omega} = \pounds_v \omega$	$\dot{\ell} = \pounds_{X_H} \ell$
momentum map	Kelvin's circulation theorem	conservation of radiance

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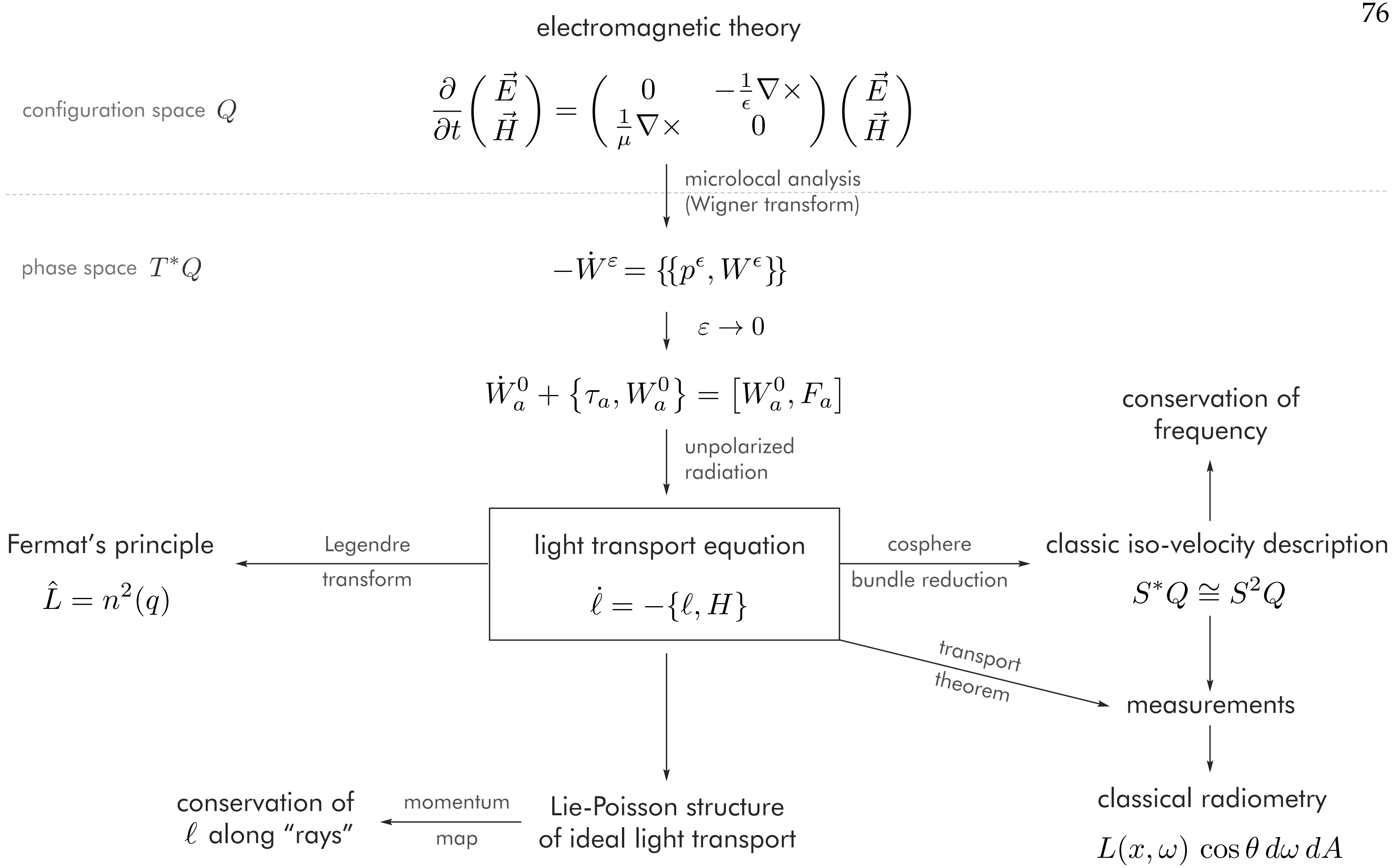
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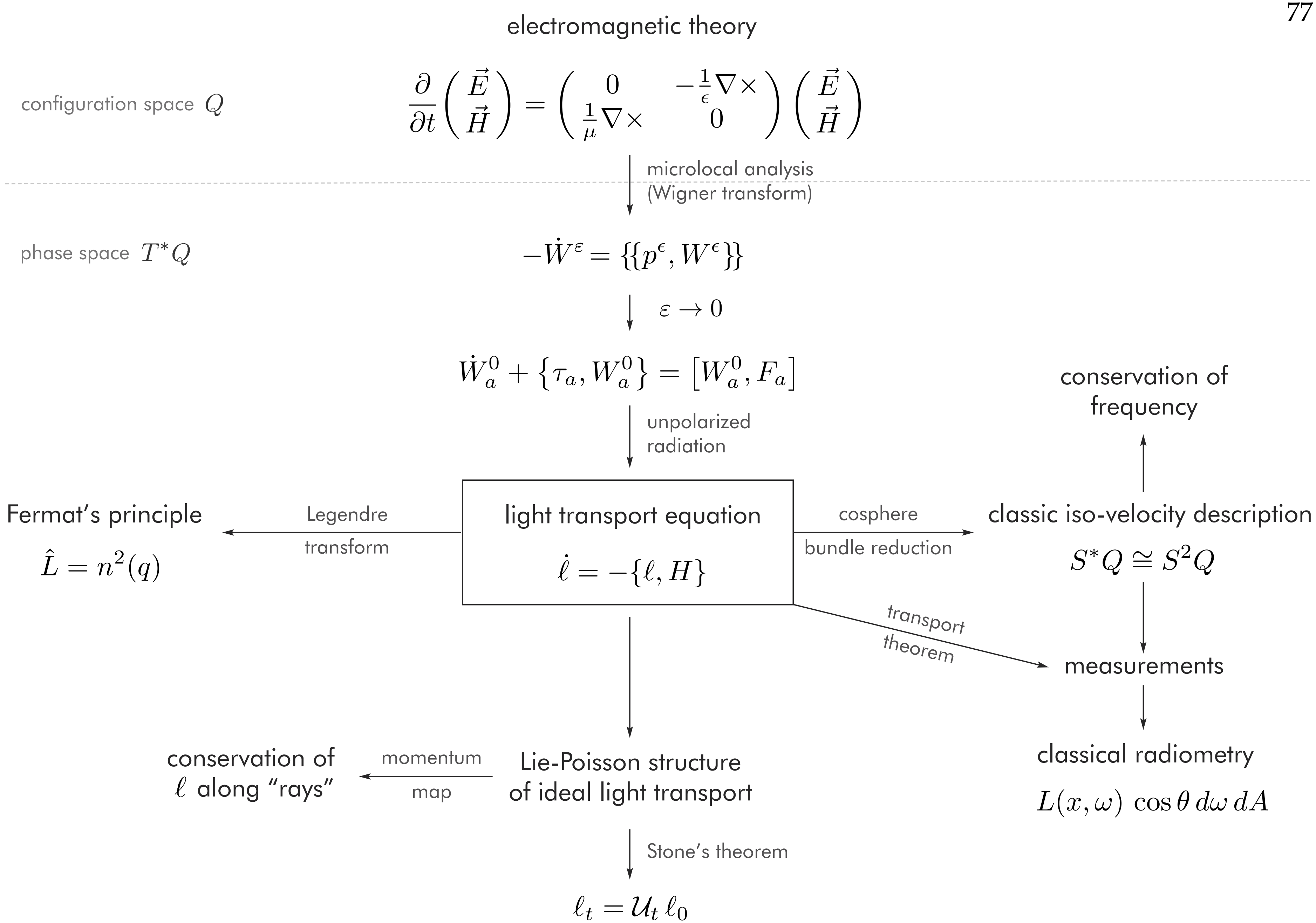
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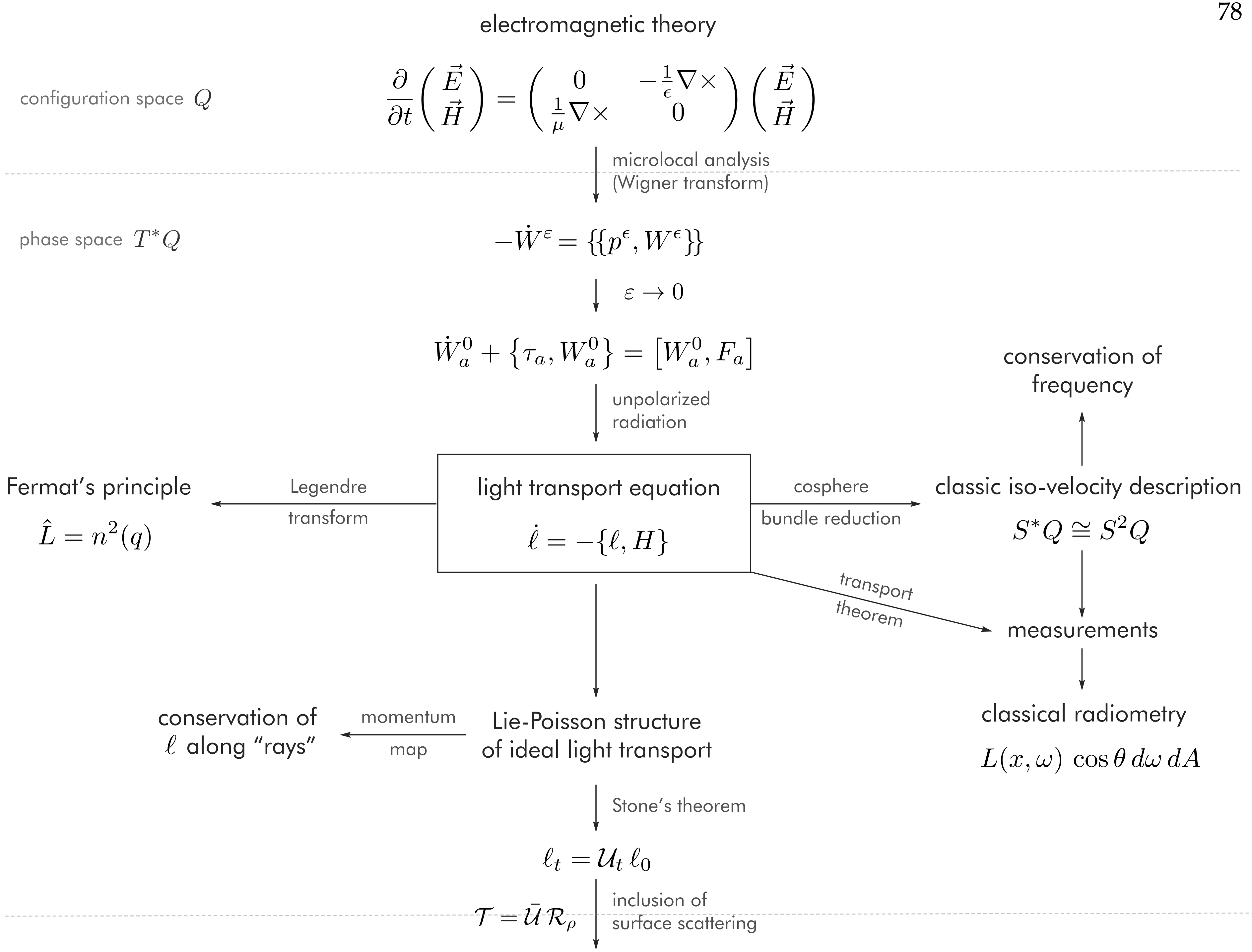
$L(x,\omega)\,\cos\theta\,d\omega\,dA$
Stone's theorem
$\ell_t = \mathcal{U}_t\,\ell_0$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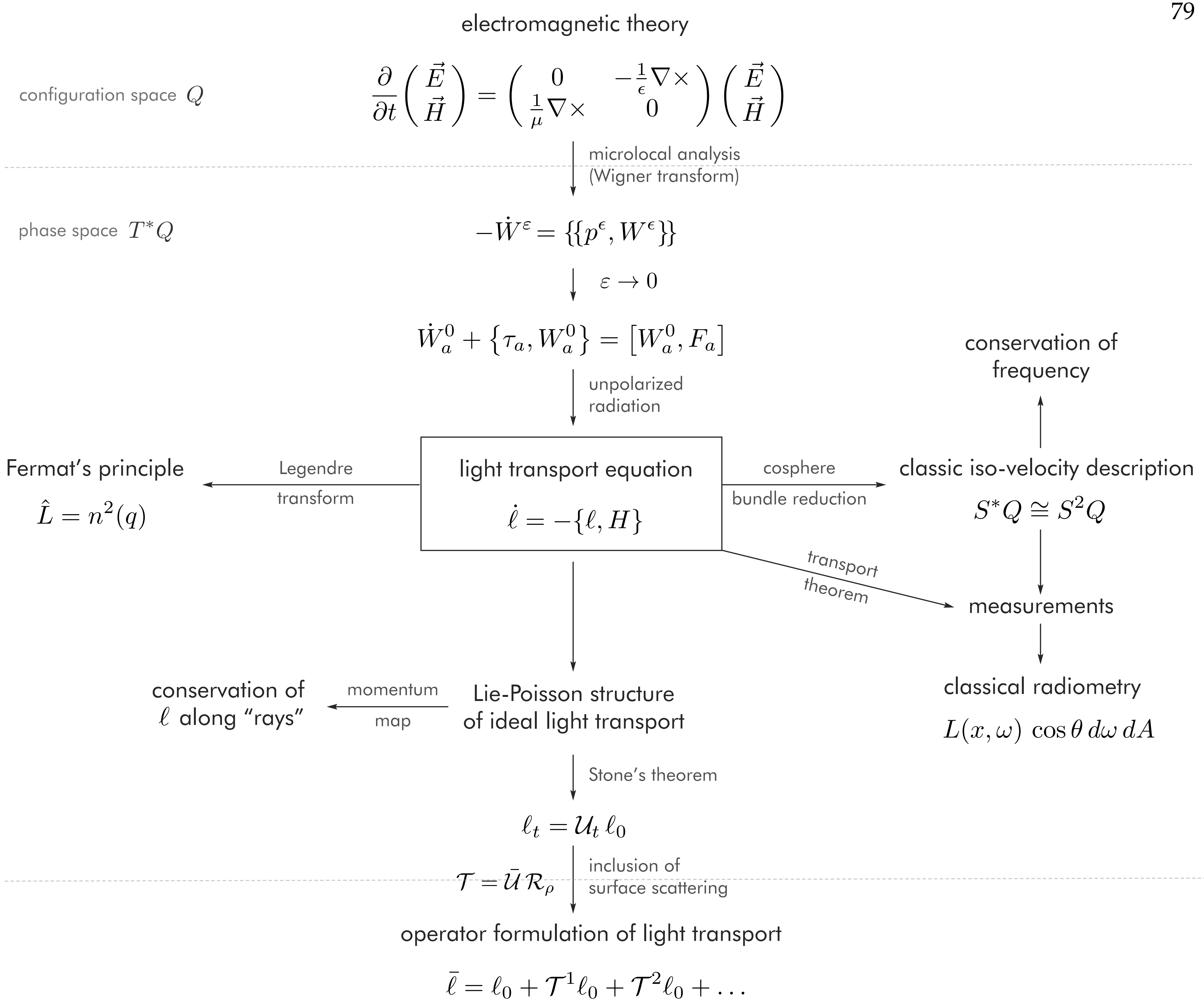
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inclusion of
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operator formulation of light transport
$\bar{\ell} = \ell_0 + \mathcal{T}^1\ell_0 + \mathcal{T}^2\ell_0 + \ldots$

... from physics
to current computations ...

Challenges

1. Visibility.

Challenges

1. Visibility.
2. Curse of dimensionality.

Challenges

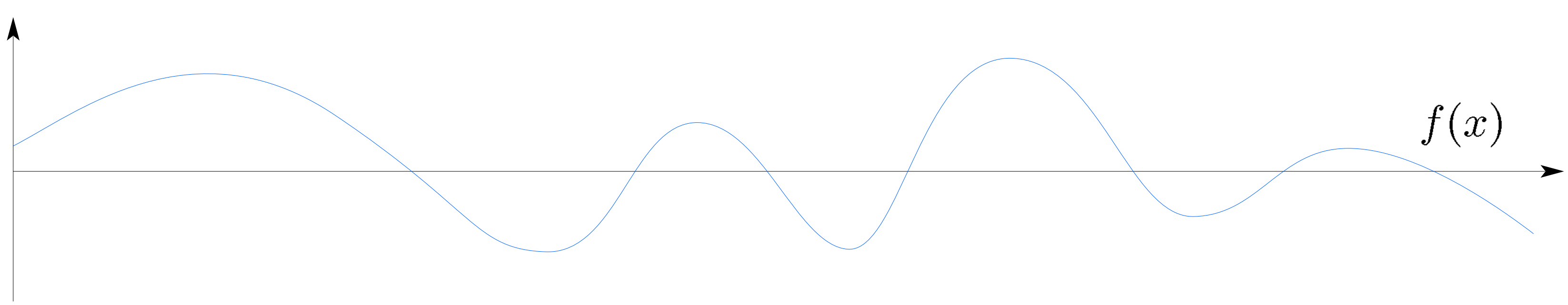
1. Visibility.
2. Curse of dimensionality.
3. Only local information is available.

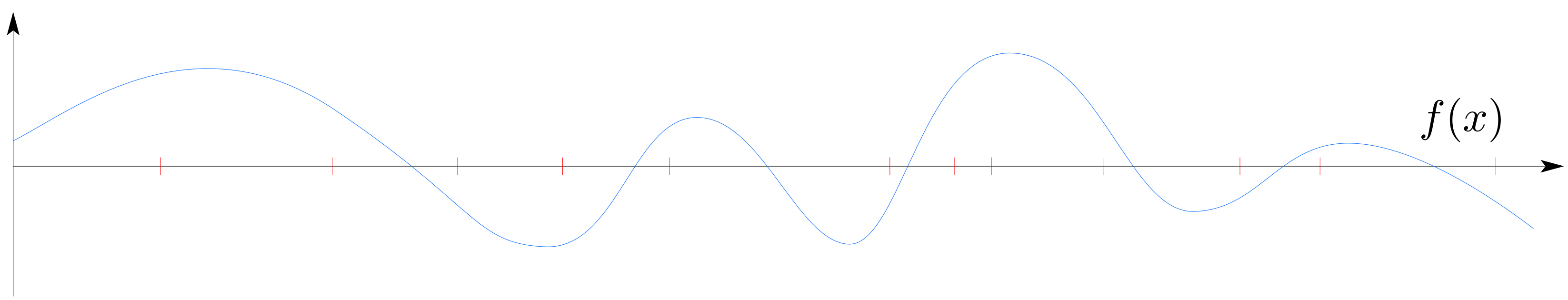
Challenges

1. Visibility.
2. Curse of dimensionality.
3. Only local information is available.

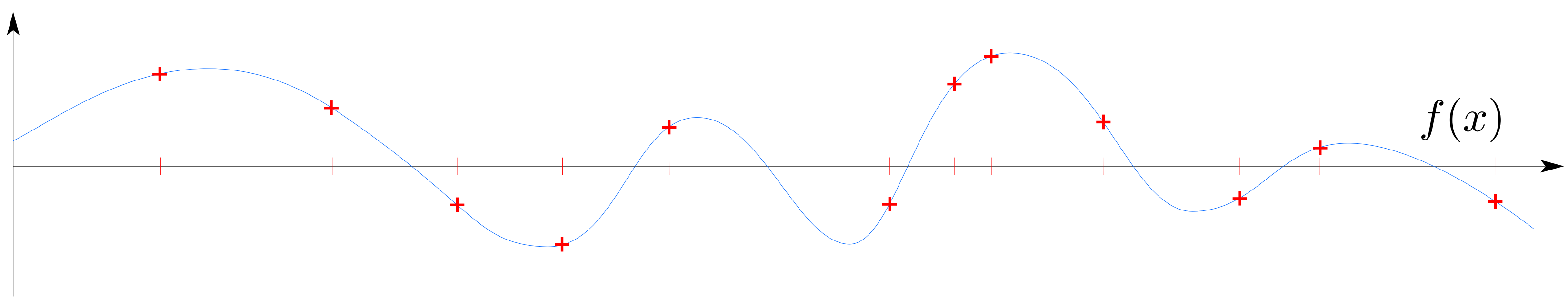
Techniques for local information

- Quadrature rules.
- Interpolation schemes.
- Sampling theorems.
- Density estimation techniques.
- (Quasi) Monte Carlo integration.
- ...

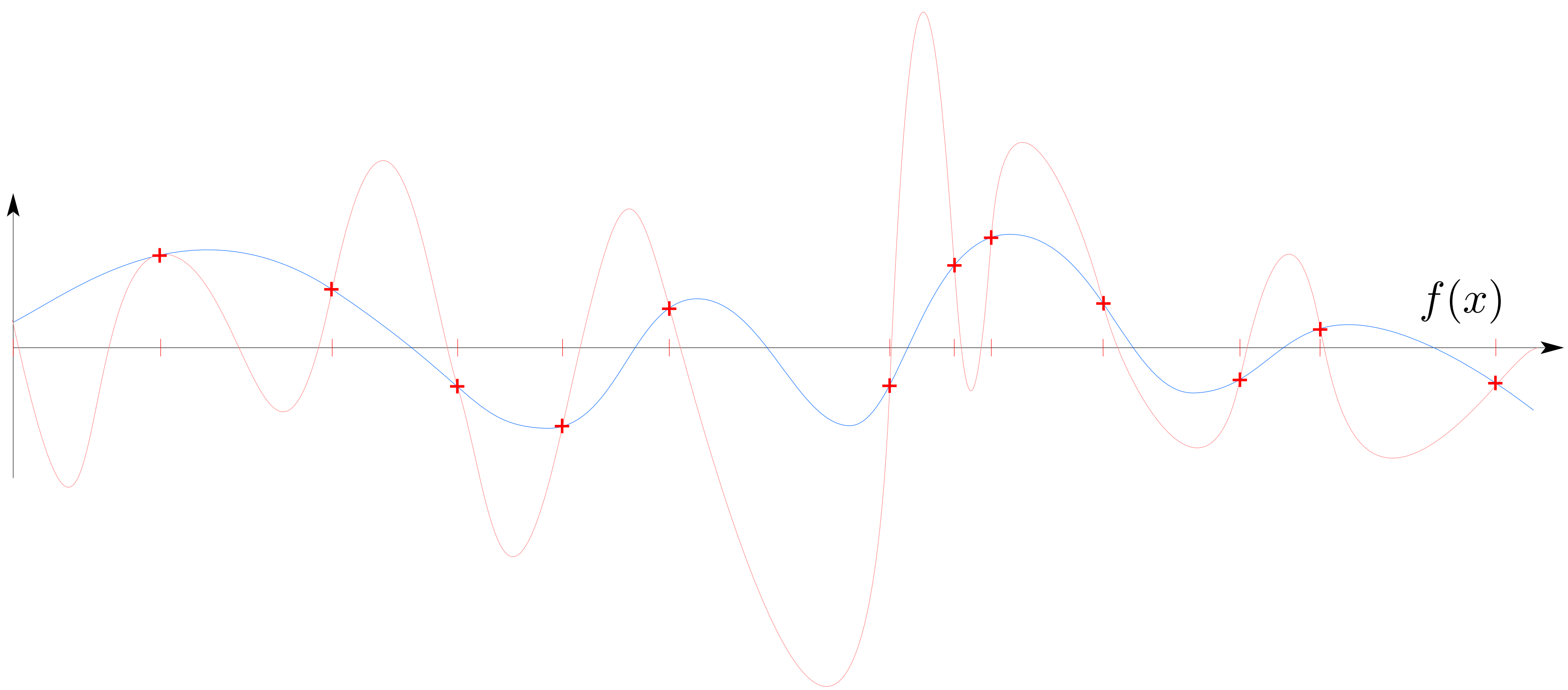

$f(x)$


$f(x)$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{f(\lambda_i)}{p(\lambda_i)} \xrightarrow[n\to\infty]{} \int_X f(x)\,dx$$

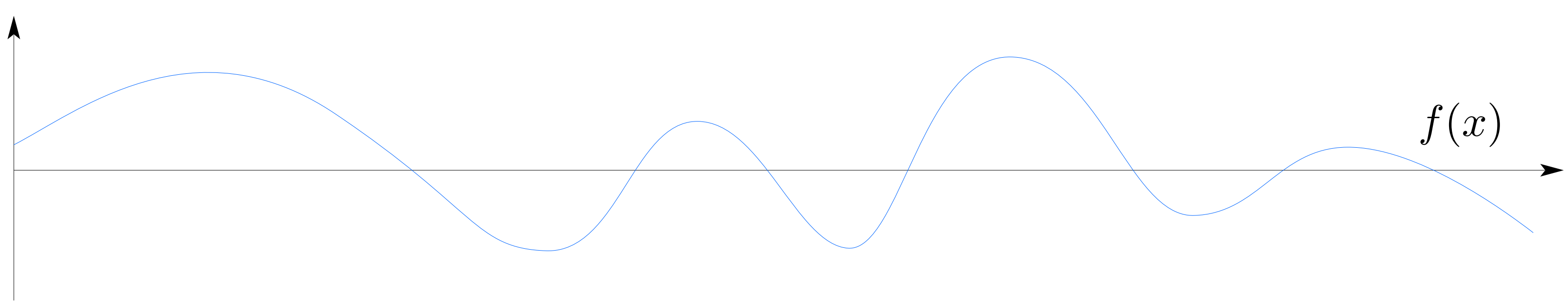


$$\bar{f}_\kappa = f(x) + \kappa \prod_{i=1}^{n} (x - \lambda_i)^2$$

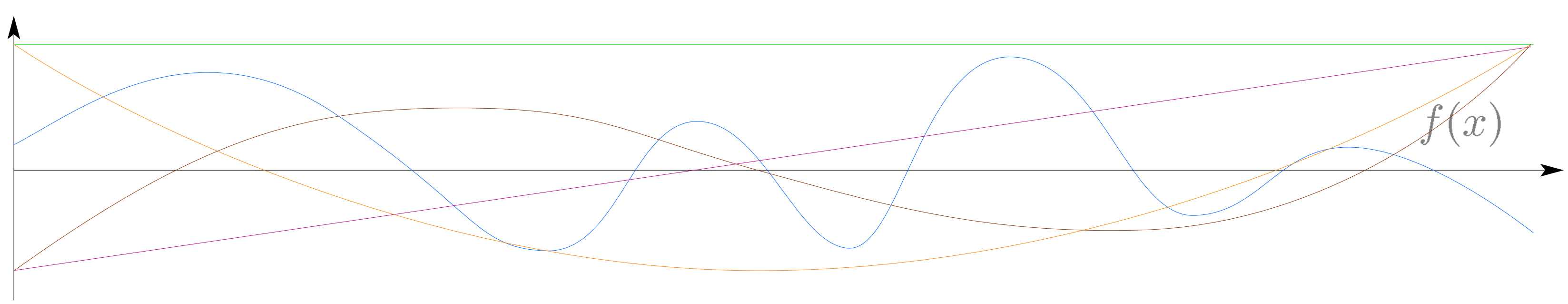


After Traub, J. F., and A. G. Werschulz, *Complexity and Information*. Cambridge University Press, 1999.

Basis expansions

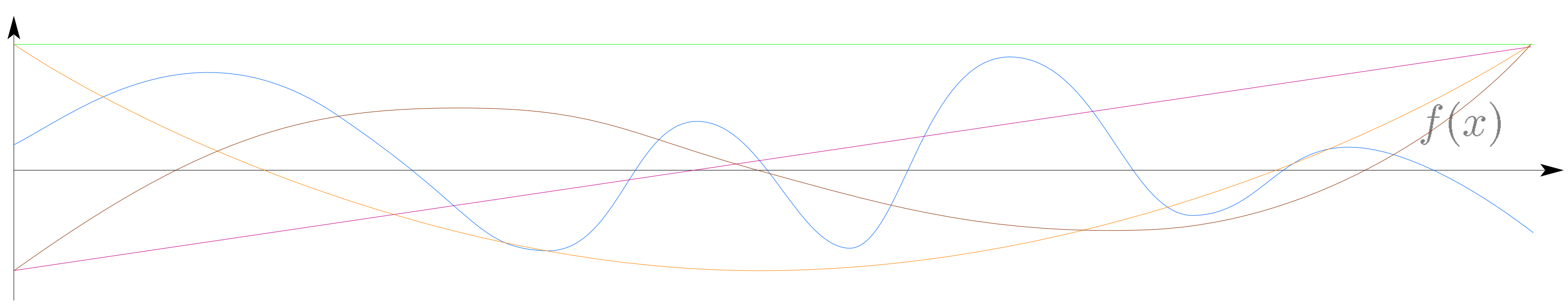


Basis expansions



Basis expansions

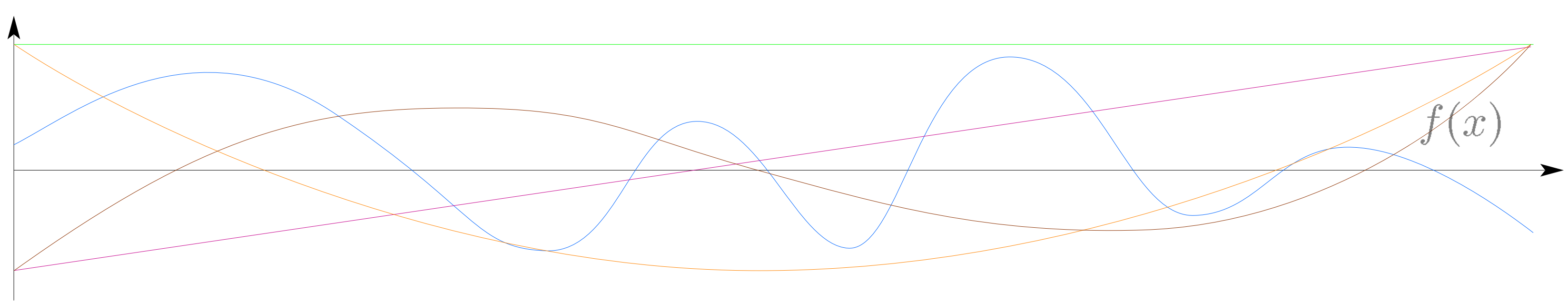
$$f(x) = \sum_{i=1}^{n} f_i \, \varphi_i(x)$$



Basis expansions

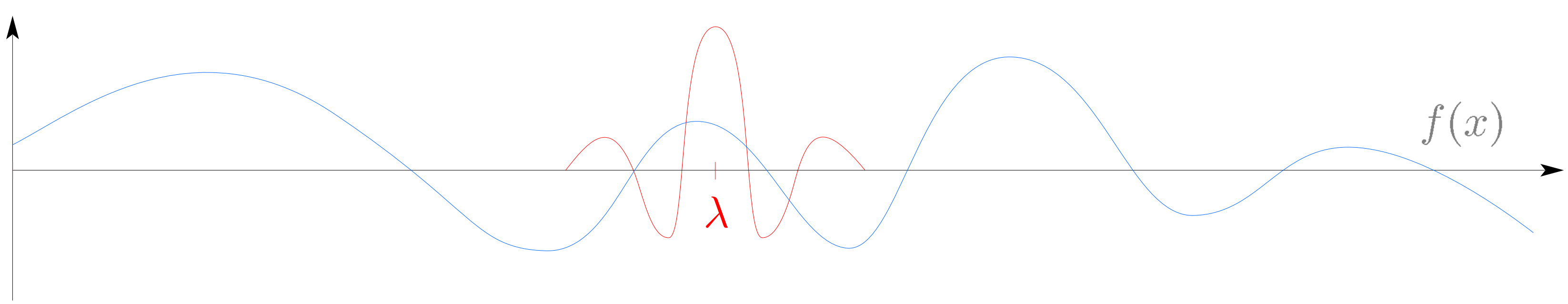
$$f(x) = \sum_{i=1}^{n} f_i \, \varphi_i(x)$$

$$= \sum_{i=1}^{n} \langle f(\bar{x}), \tilde{\varphi}_i(\bar{x}) \rangle \, \varphi_i(x)$$



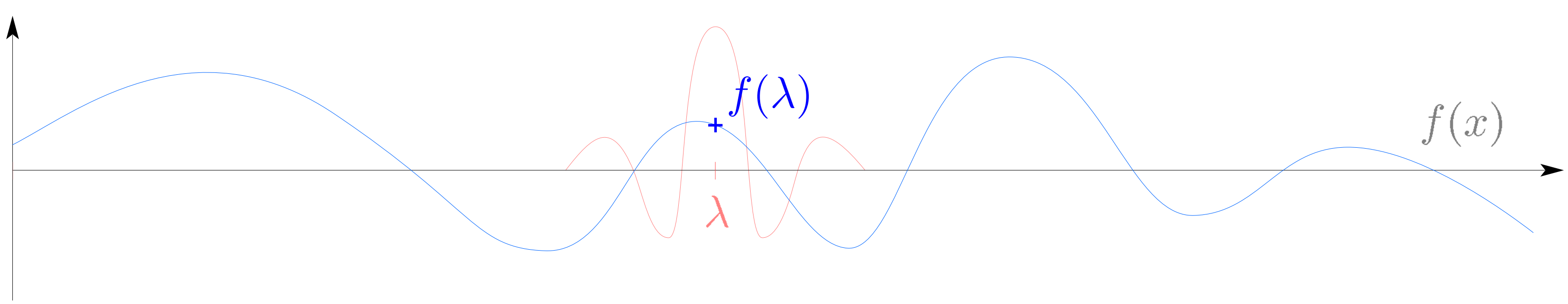
Reproducing kernel

$$k_\lambda(\bar{x})$$



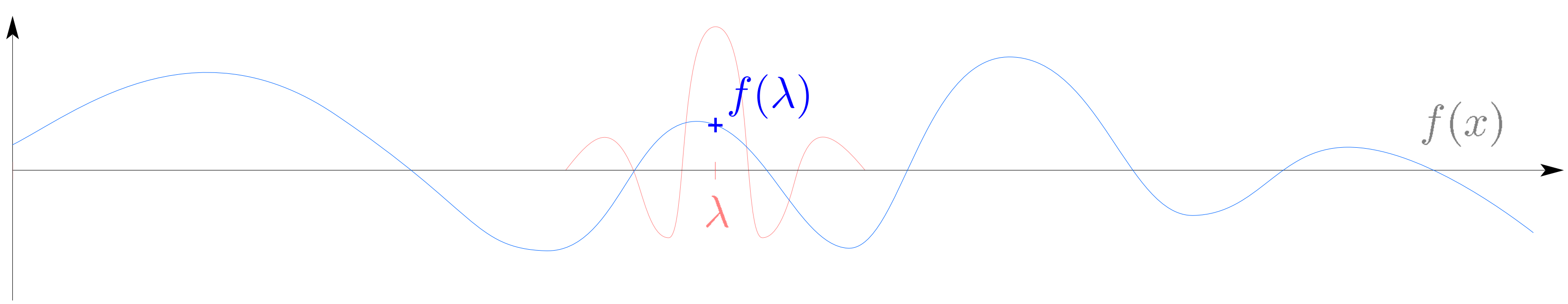
Reproducing kernel

$$f(\lambda) = \langle f(\bar{x}), k_\lambda(\bar{x}) \rangle$$



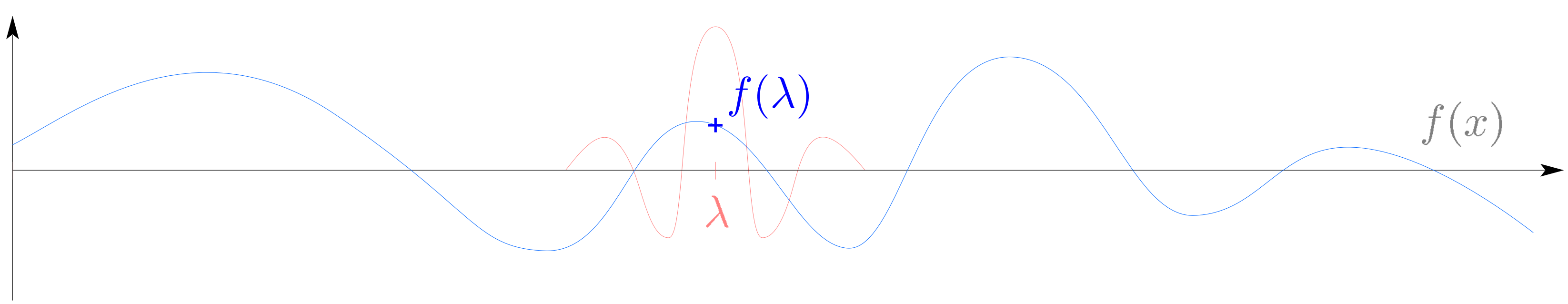
Reproducing kernel

$$f(\lambda) = \langle f(\bar{x}), k_\lambda(\bar{x}) \rangle = \delta_\lambda(f)$$



Reproducing kernel

$$k_\lambda(x) = k(\lambda, x) = \sum_{i=1}^{n} \phi_i(\lambda)\, \phi_i(x)$$



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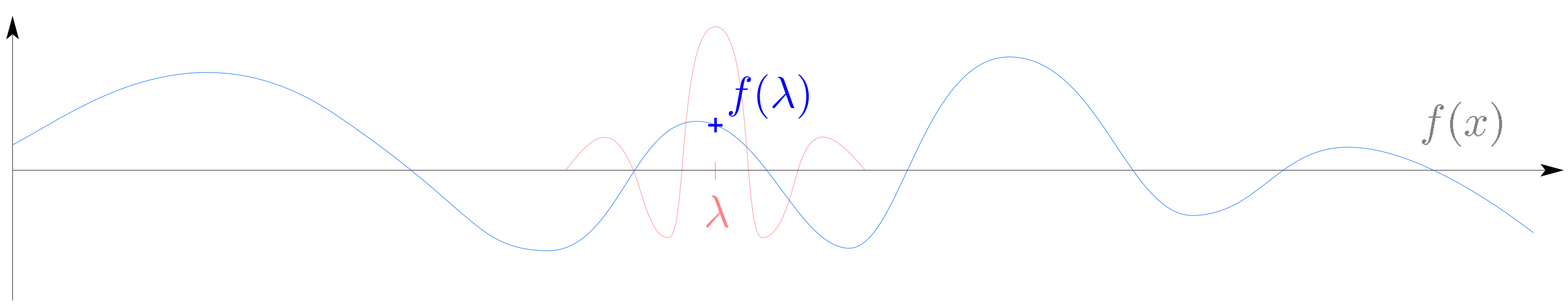
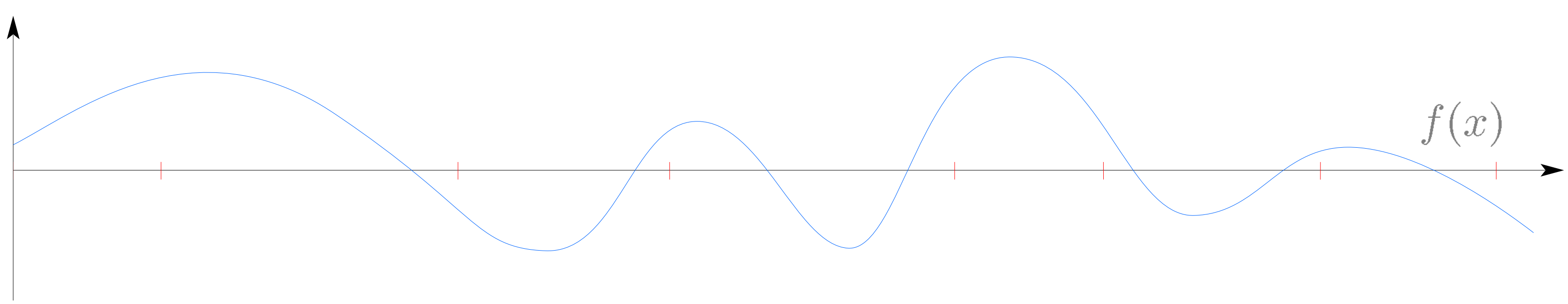
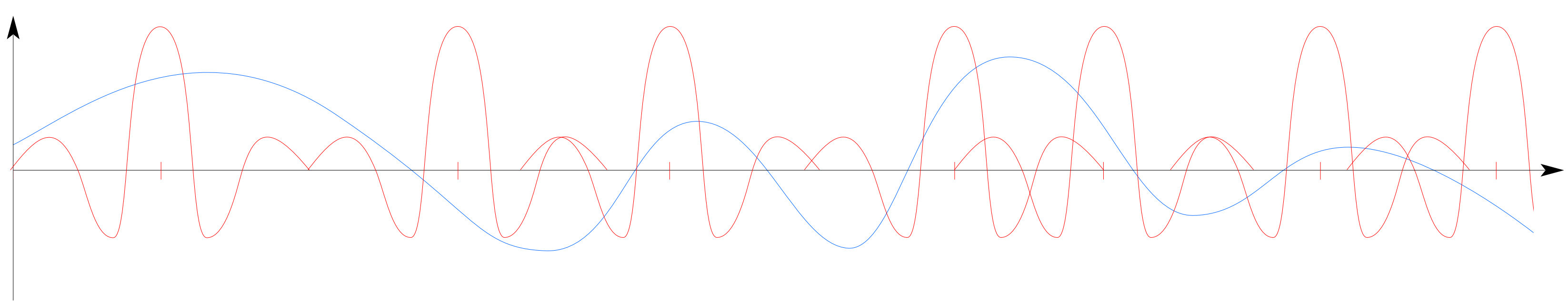


Reproducing kernel bases



Reproducing kernel bases

$$f(x) = \sum_{i=1}^{n} \langle f(\bar{x}), k_{\lambda_i}(\bar{x}) \rangle \, \tilde{k}_i(x)$$



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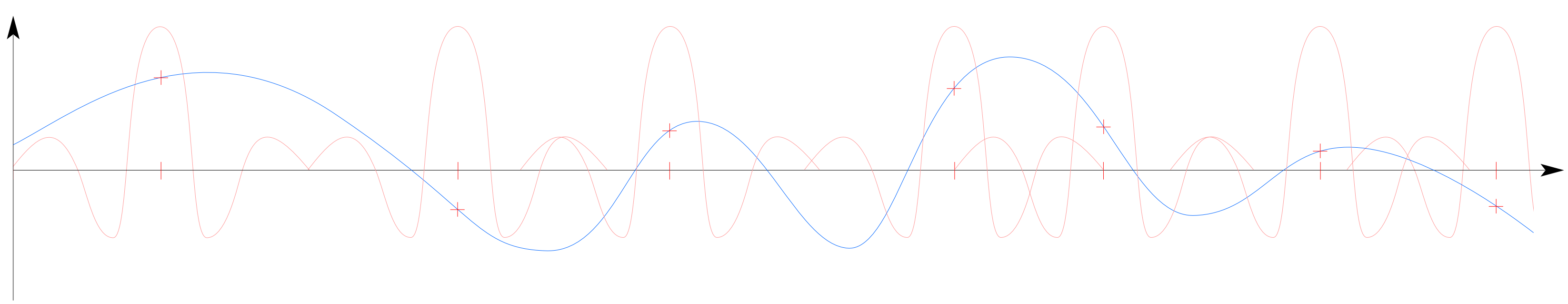

Reproducing kernel bases

- Shannon sampling theorem.
- Gauss-Legendre quadrature.
- Lagrange interpolation.
- Monte Carlo integration.
- ...

Integration

$$I = \int_X f(x)\,dx$$

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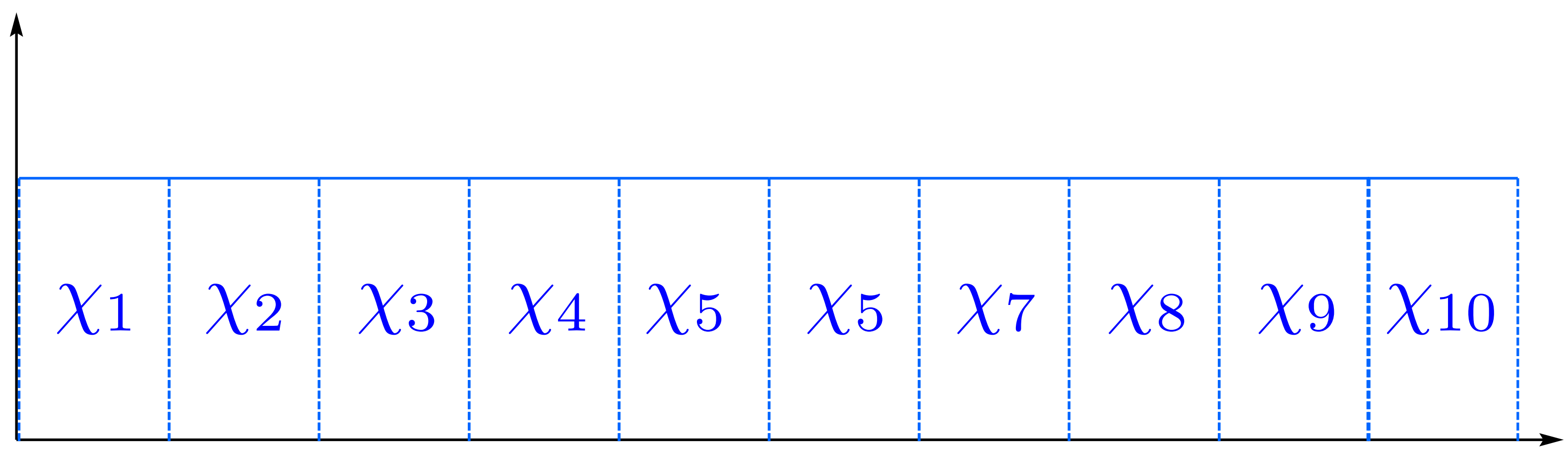

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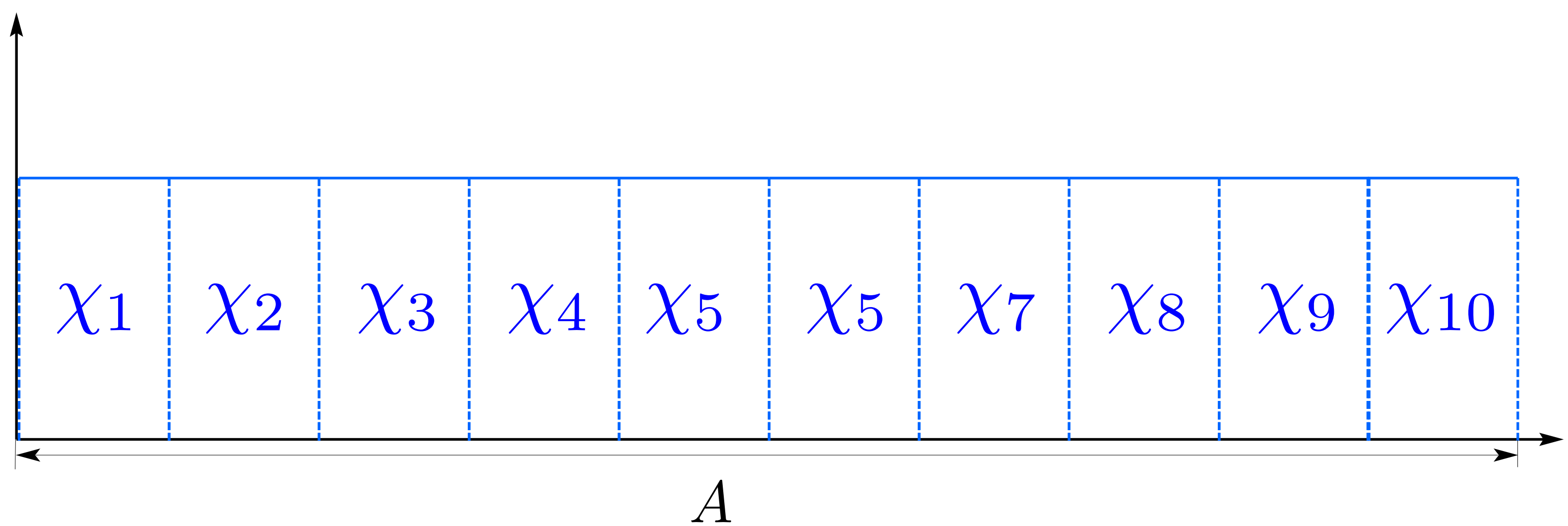


Integration

$$
\begin{aligned}
\int_X f(x)\,dx &= \sum_{i=1}^{m} w_i\, f(\lambda_i) \\
&= \sum_{i=1}^{n} \left(\int_X \chi_i\, dx \right) f(\lambda_i)
\end{aligned}
$$

Integration

$$\int_X f(x)\,dx = \sum_{i=1}^{m} w_i\, f(\lambda_i)$$

$$= \sum_{i=1}^{n} \left(\int_X \chi_i\, dx \right) f(\lambda_i)$$

$$= \frac{\mu(A)}{n} \sum_{i=1}^{n} f(\lambda_i)$$

Reproducing kernel bases

1. Arbitrary domains and function spaces.

Reproducing kernel bases

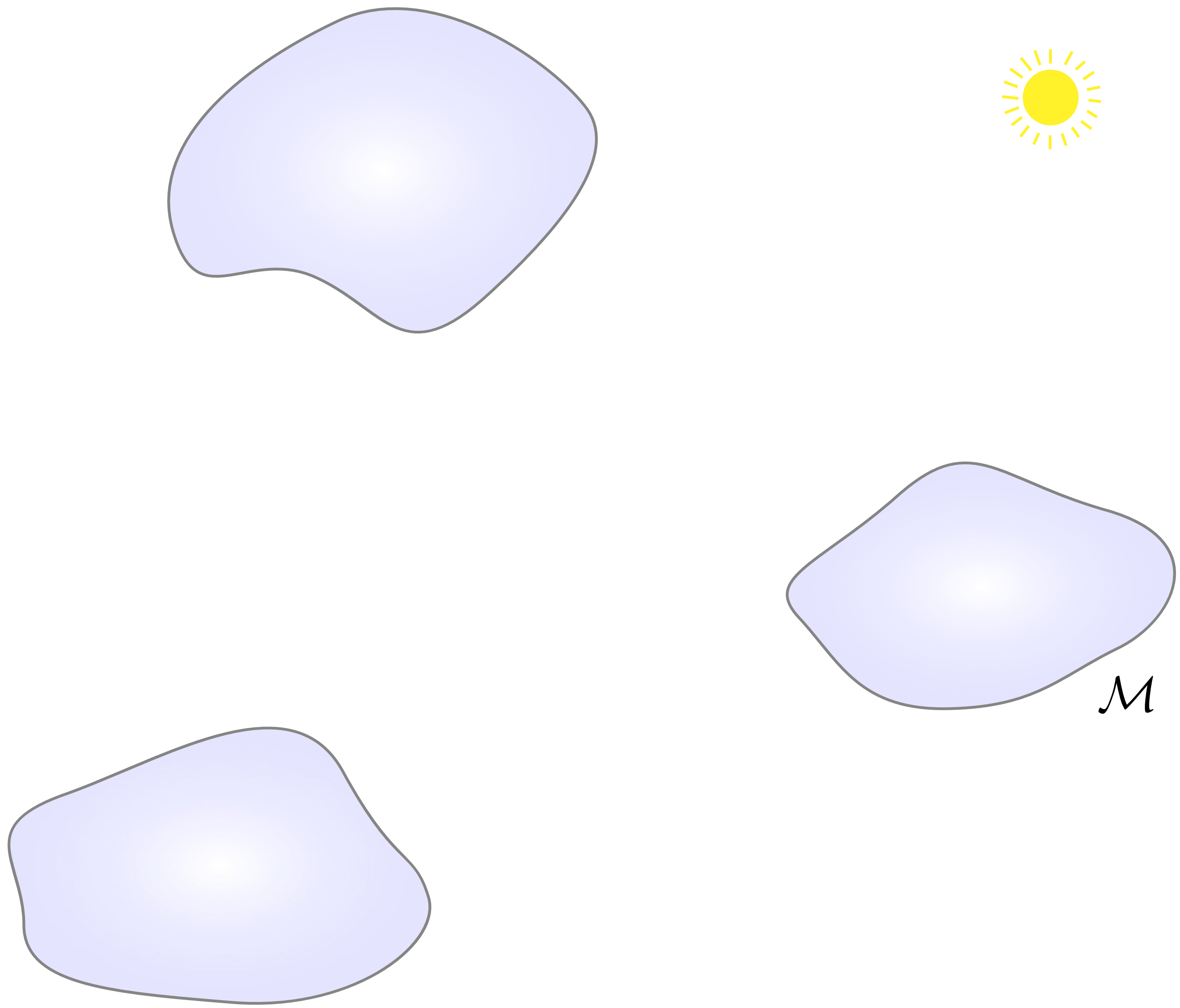
1. Arbitrary domains and function spaces.
2. Optimization of reproducing points.

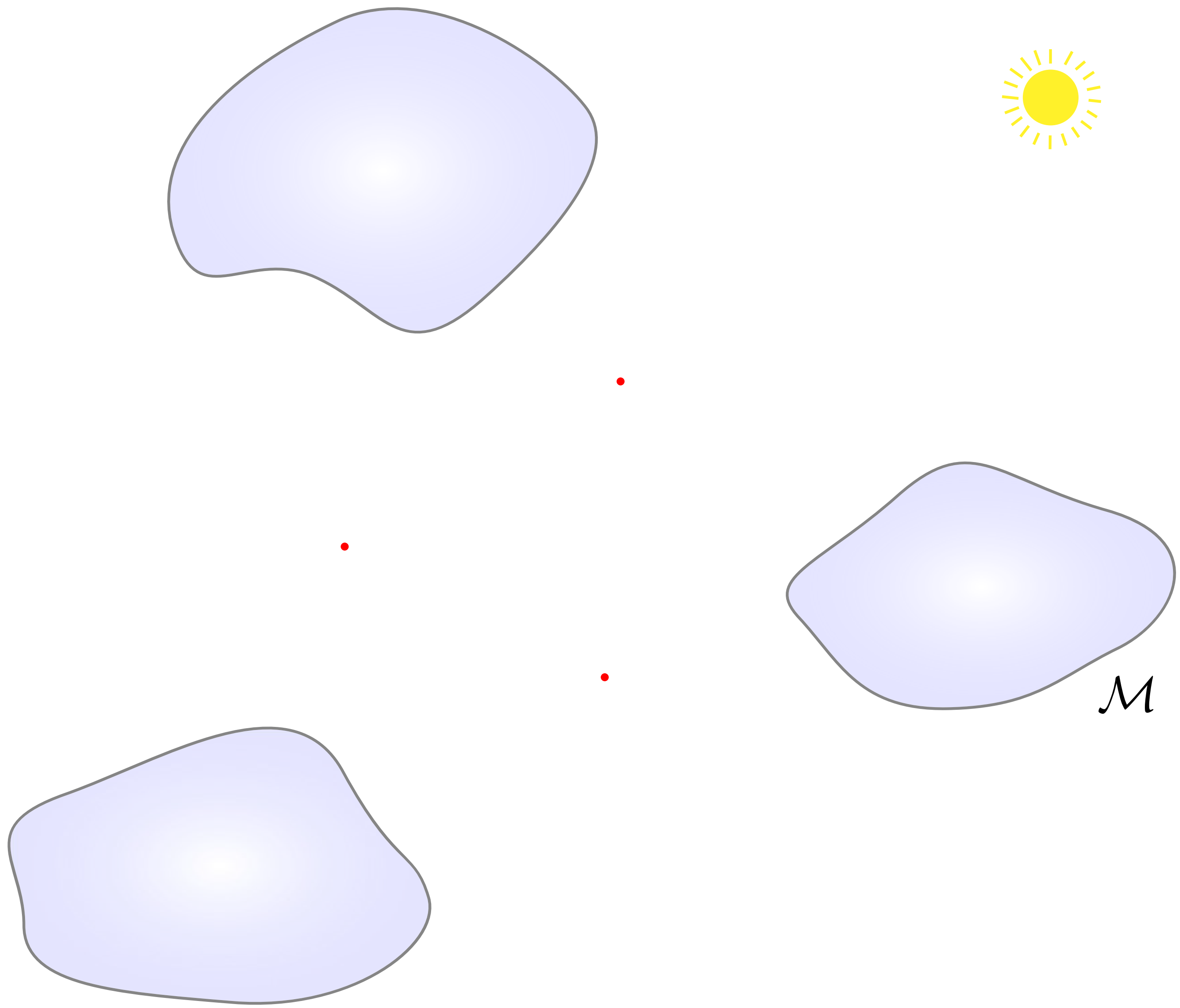
Reproducing kernel bases

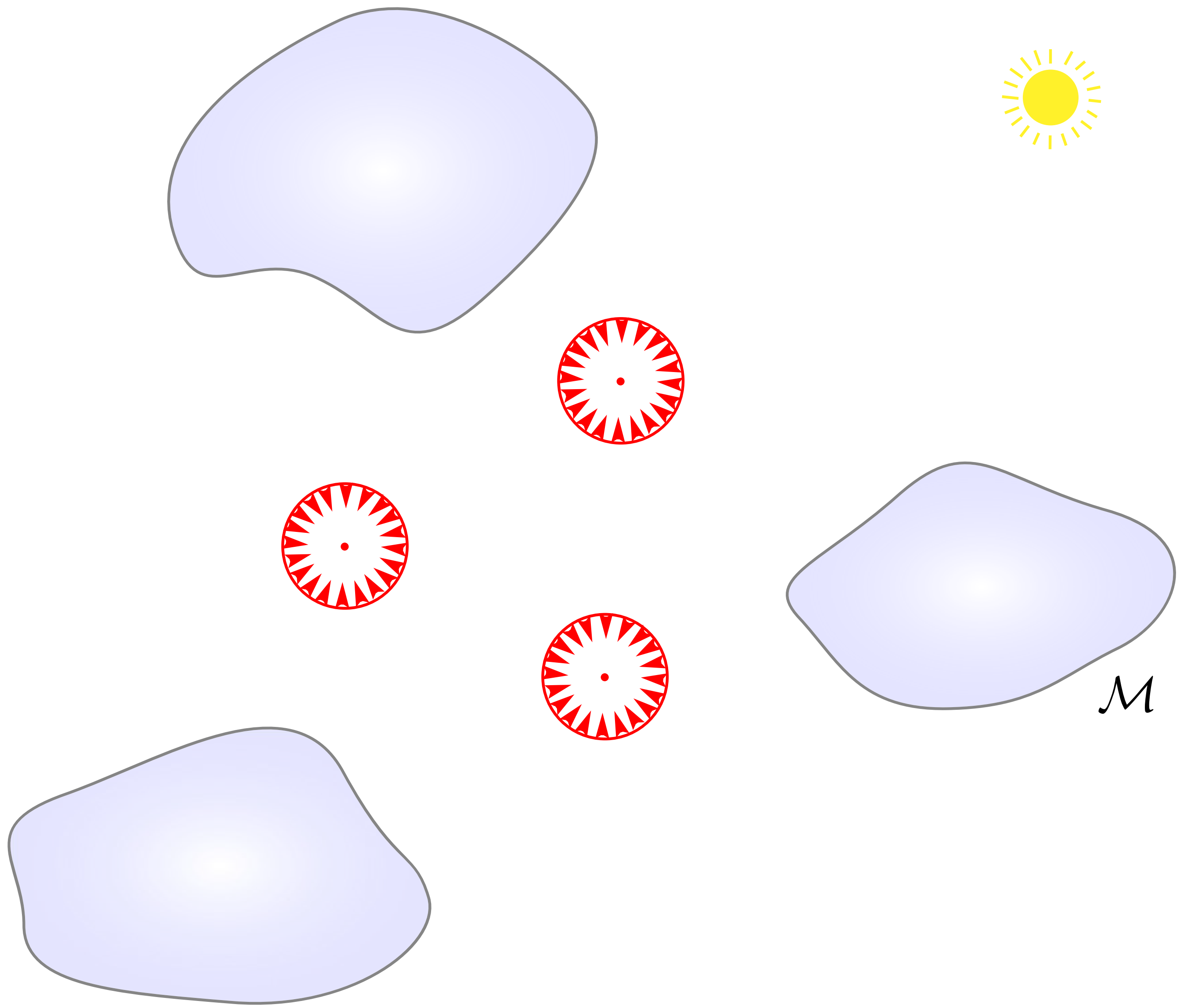
1. Arbitrary domains and function spaces.
2. Optimization of reproducing points.
3. Overcomplete representations.

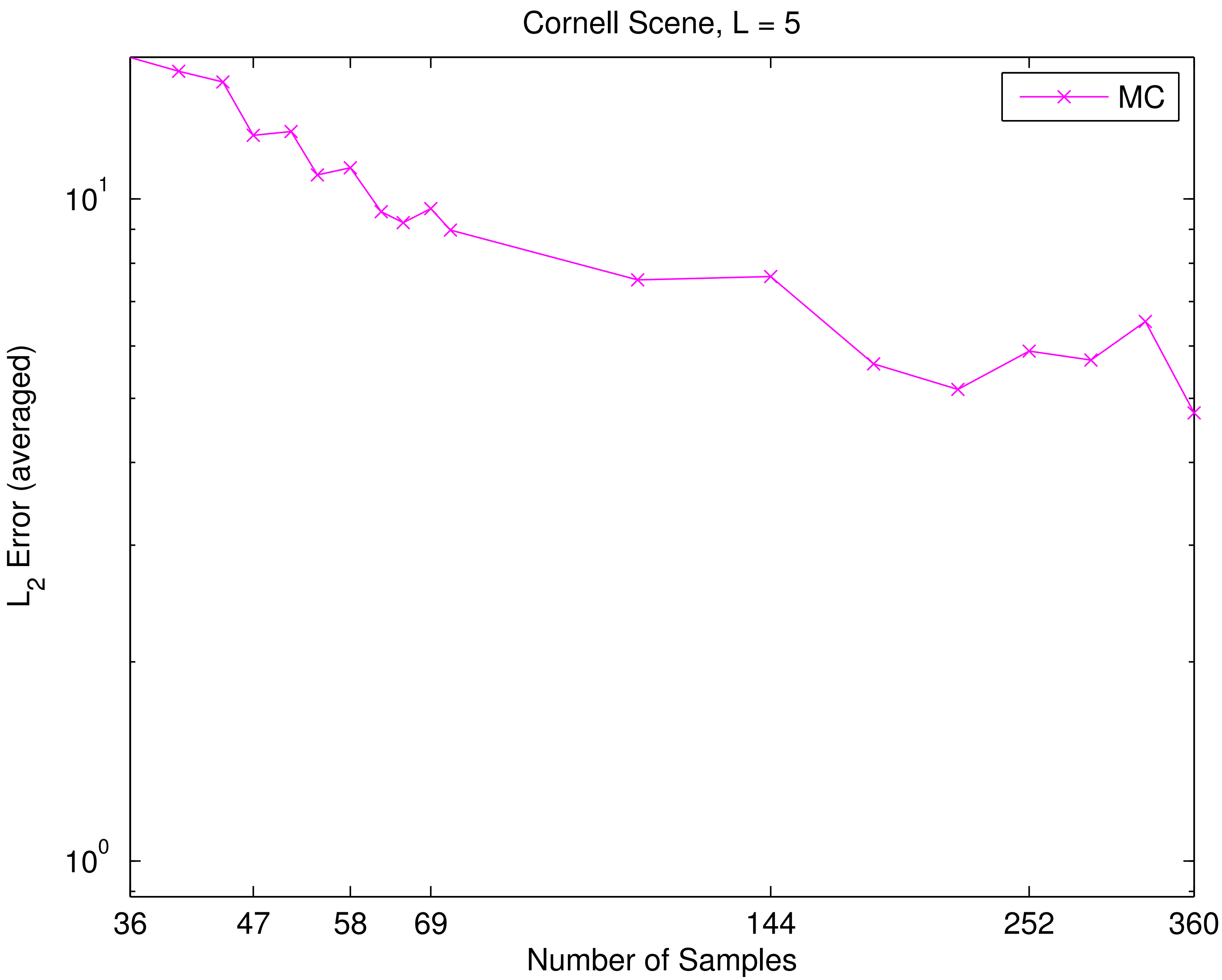
Reproducing kernel bases

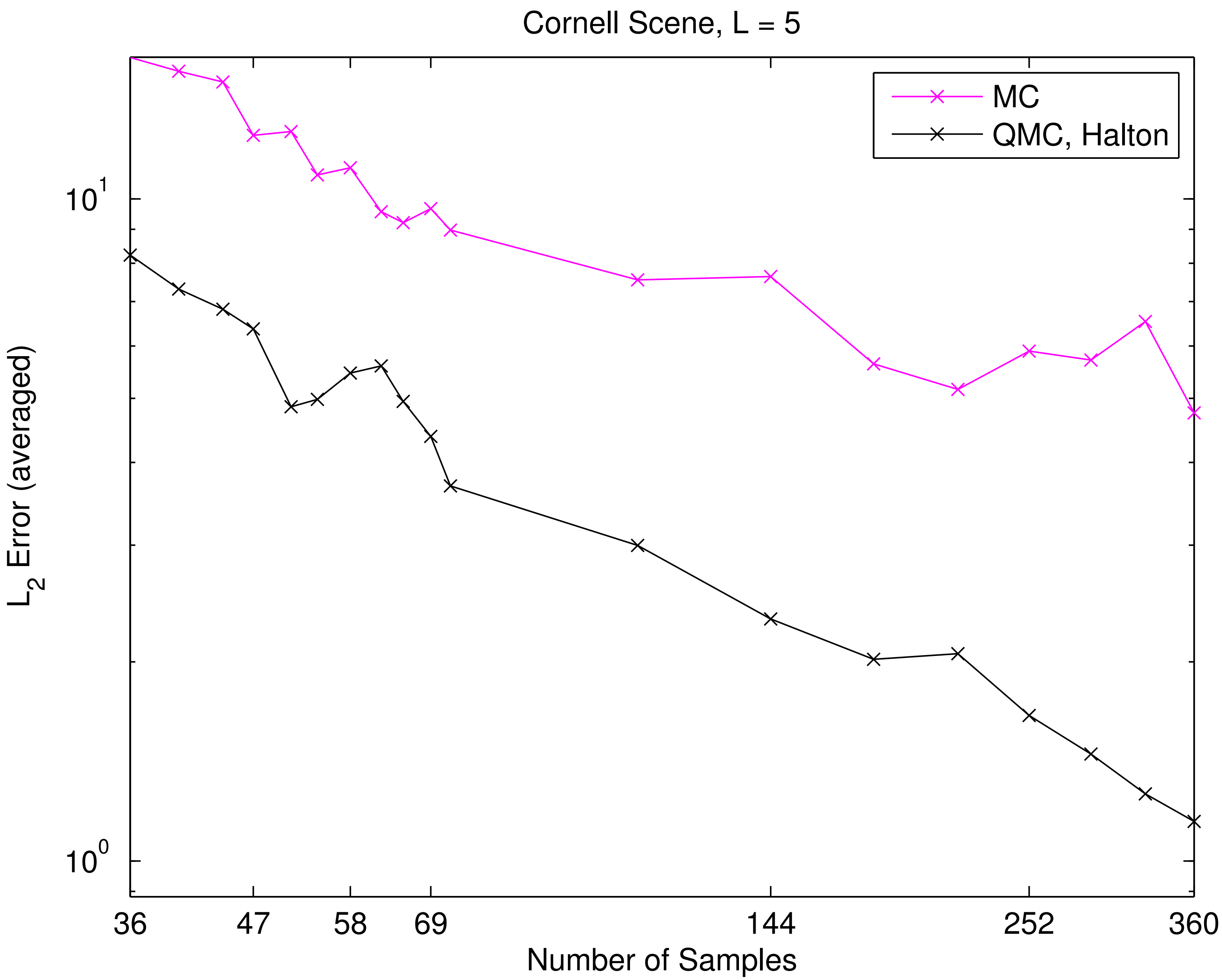
1. Arbitrary domains and function spaces.
2. Optimization of reproducing points.
3. Overcomplete representations.
4. Error analysis.

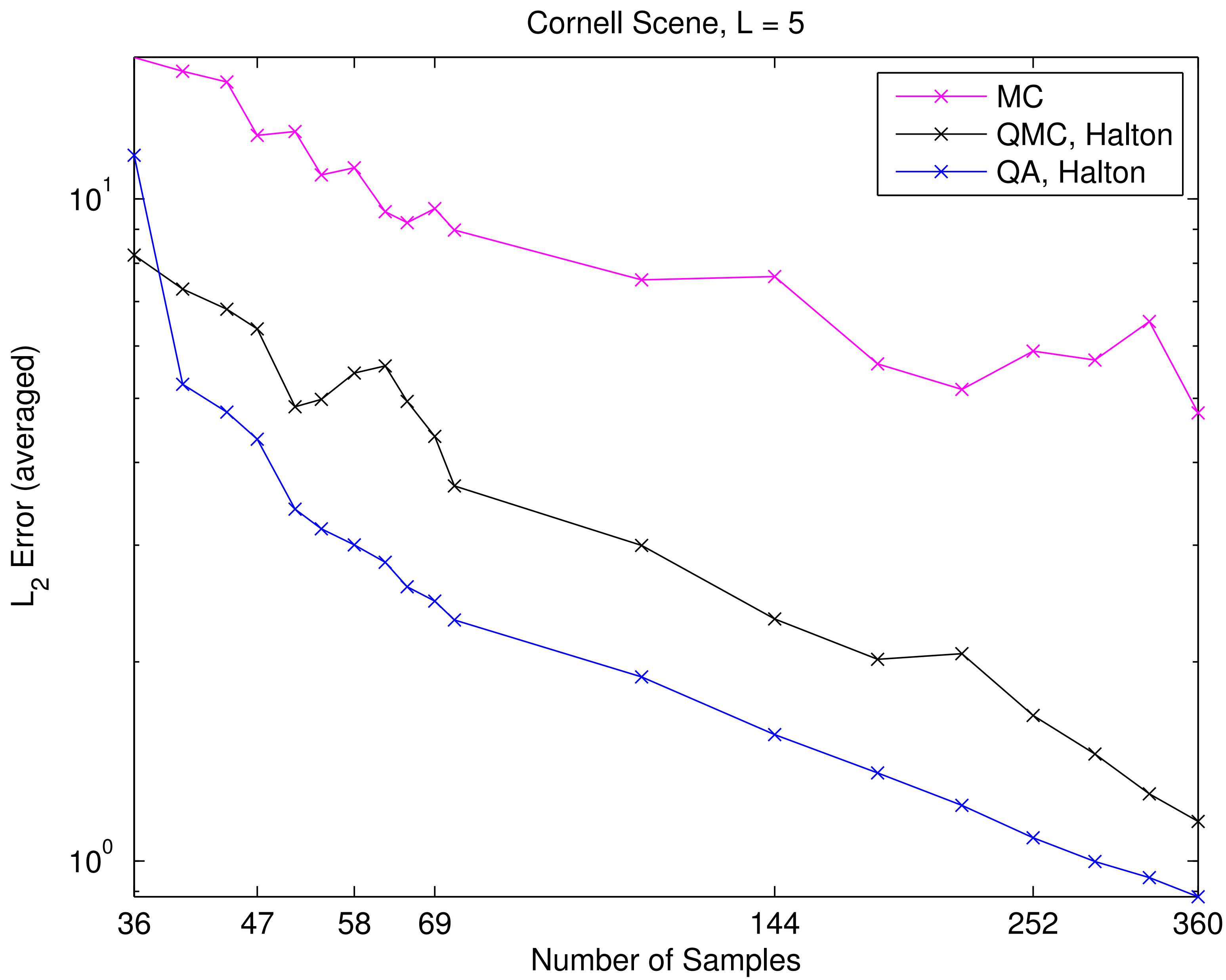

$\mathcal{M}$

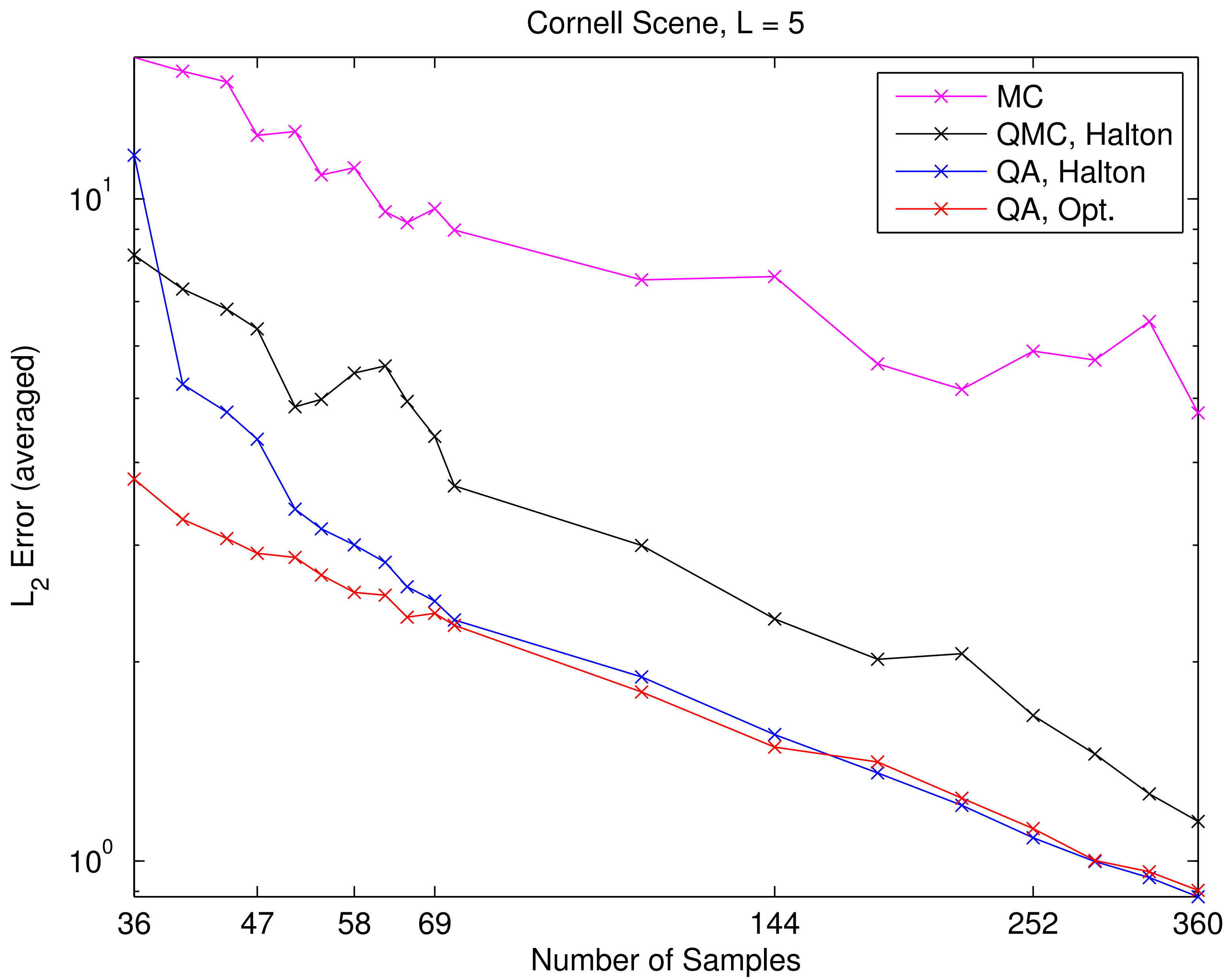

$\mathcal{M}$

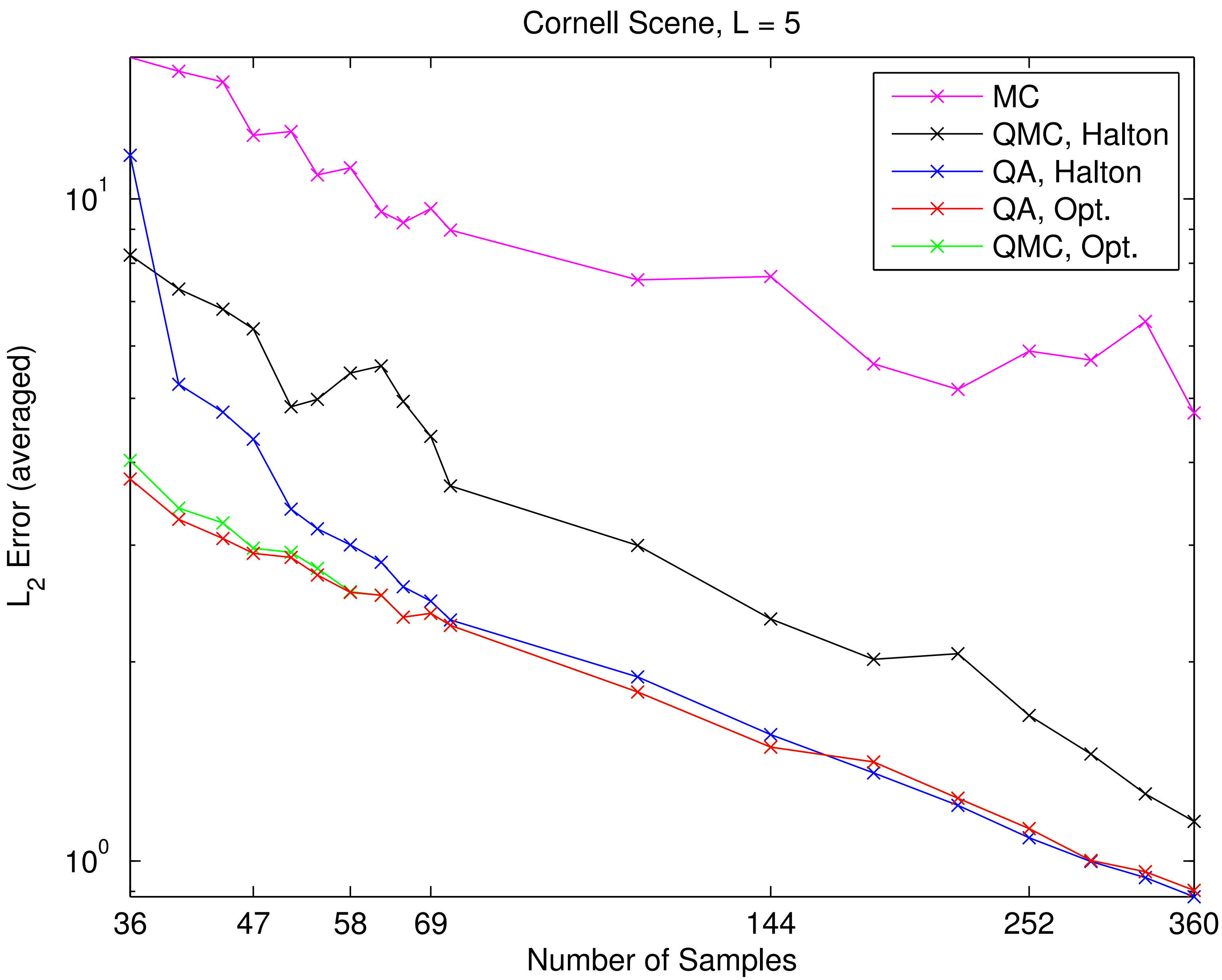

$\mathcal{M}$


Cornell Scene, L = 5
MC
10^1
10^0
L_2 Error (averaged)
36
47
58
69
144
252
360
Number of Samples


Cornell Scene, L = 5
MC
QMC, Halton
10^1
10^0
L_2 Error (averaged)
36
47
58
69
144
252
360
Number of Samples


Cornell Scene, L = 5
MC
QMC, Halton
QA, Halton
10^1
10^0
L_2 Error (averaged)
36
47
58
69
144
252
360
Number of Samples


Cornell Scene, L = 5
MC
QMC, Halton
QA, Halton
QA, Opt.
10^1
10^0
L_2 Error (averaged)
36
47
58
69
144
252
360
Number of Samples


Cornell Scene, L = 5
MC
QMC, Halton
QA, Halton
QA, Opt.
QMC, Opt.
L_2 Error (averaged)
10^1
10^0
36
47
58
69
144
252
360
Number of Samples

Reproducing kernel bases

1. Arbitrary domains and function spaces.
2. Optimization of reproducing points.
3. Overcomplete representations.
4. Error analysis.

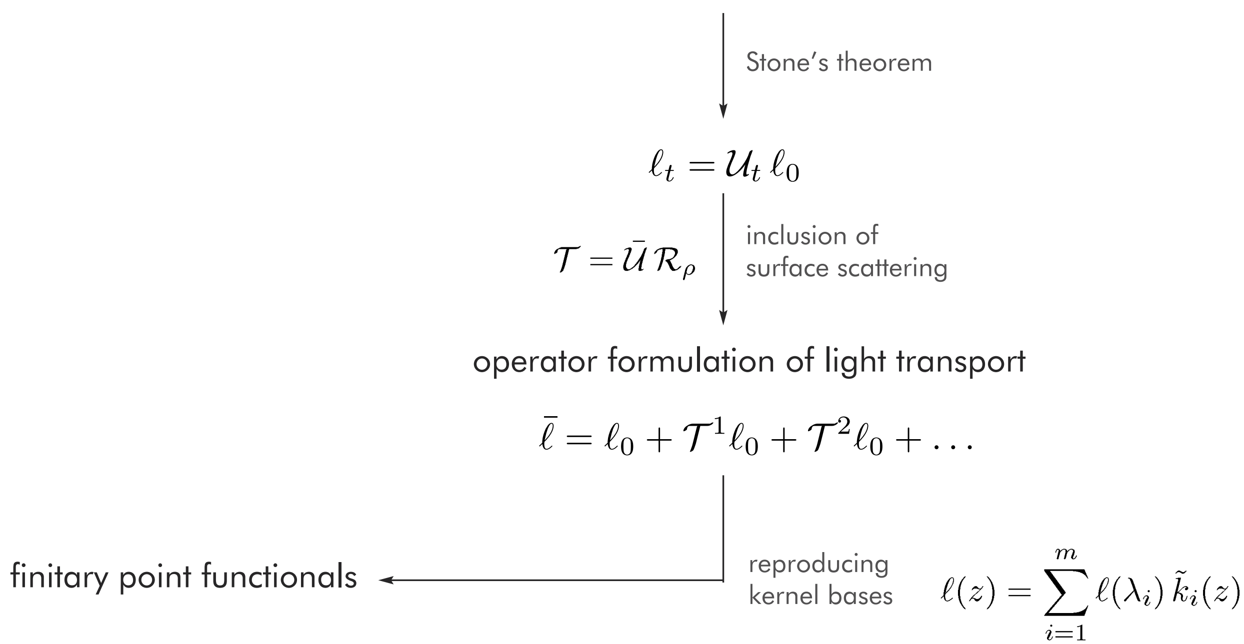
↓ Stone's theorem

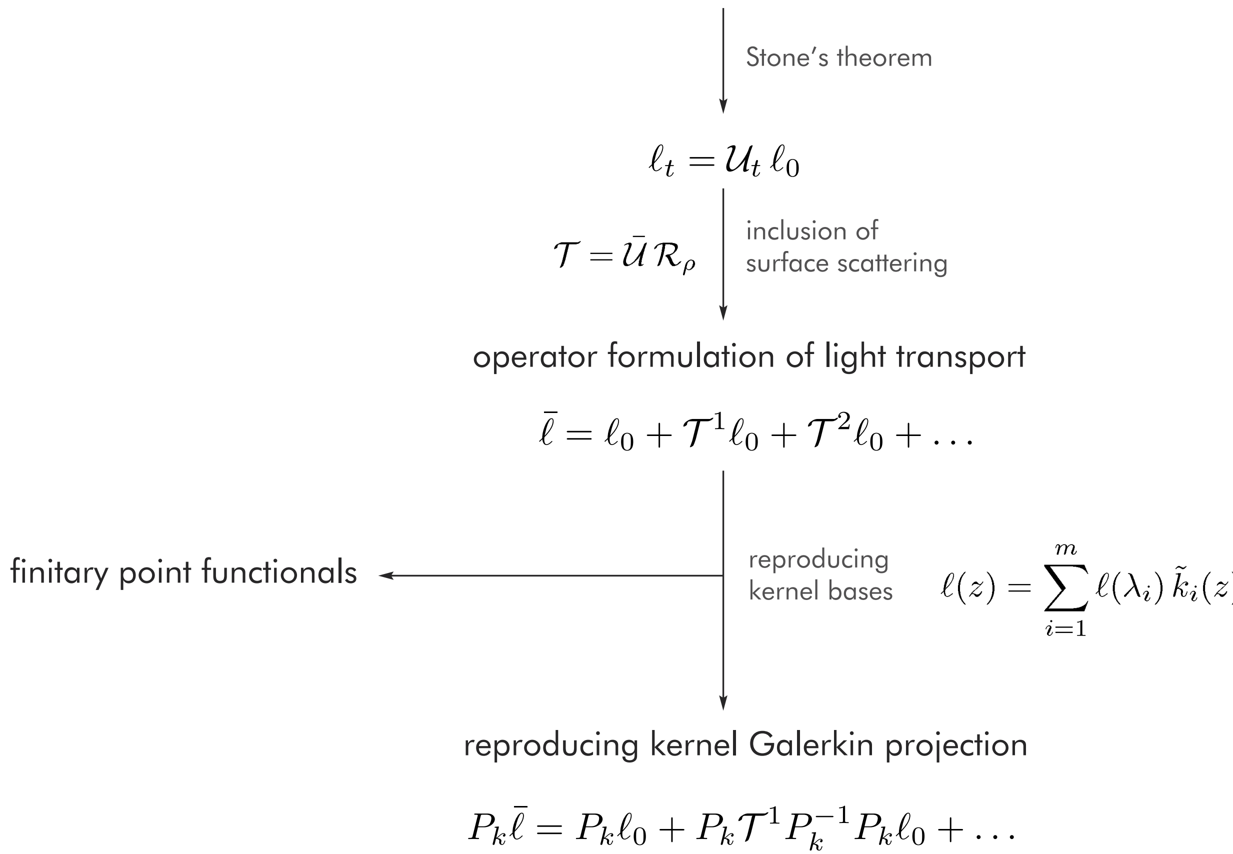
$$\ell_t = \mathcal{U}_t \, \ell_0$$

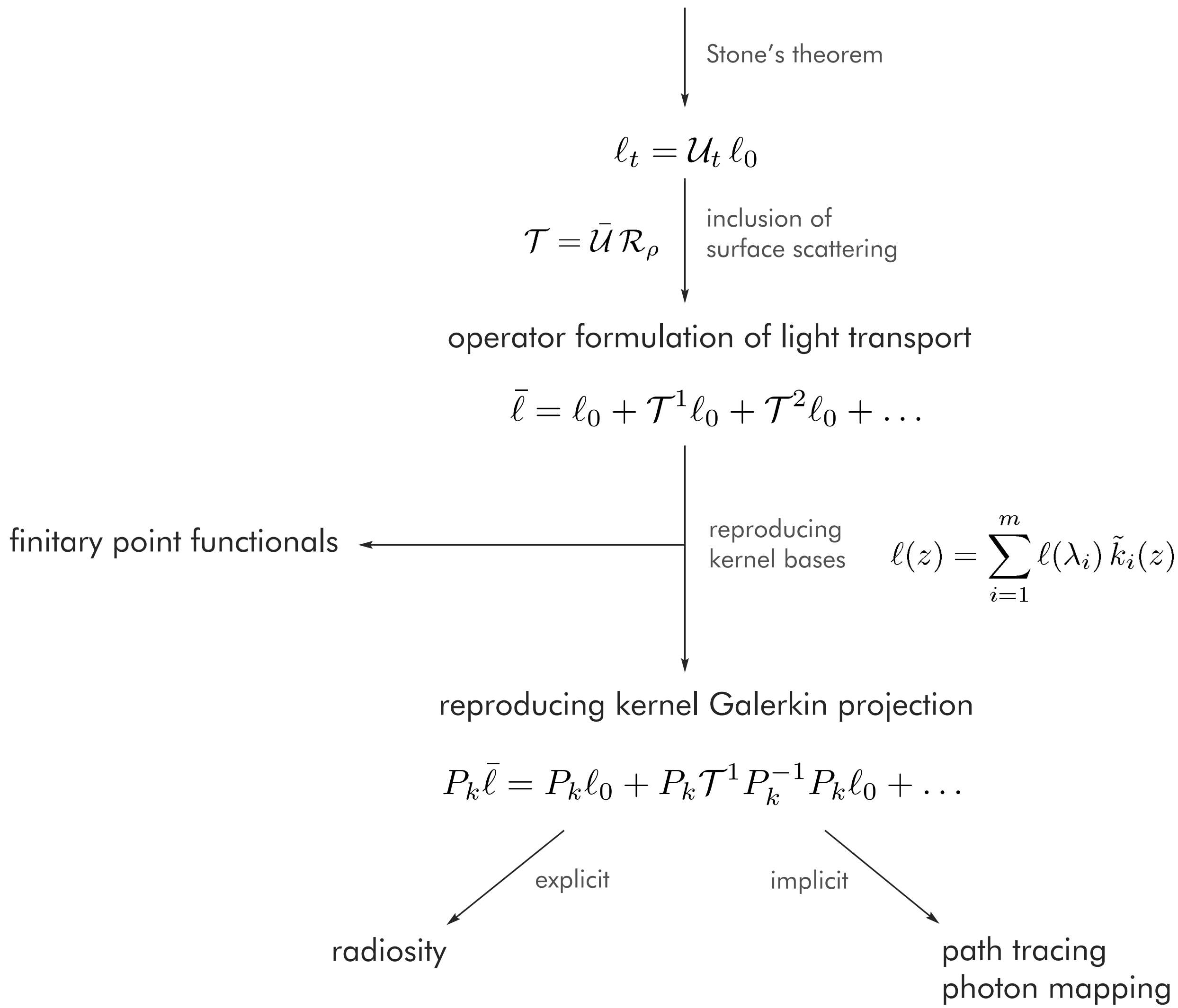
$\mathcal{T} = \bar{\mathcal{U}} \, \mathcal{R}_\rho$ ↓ inclusion of surface scattering

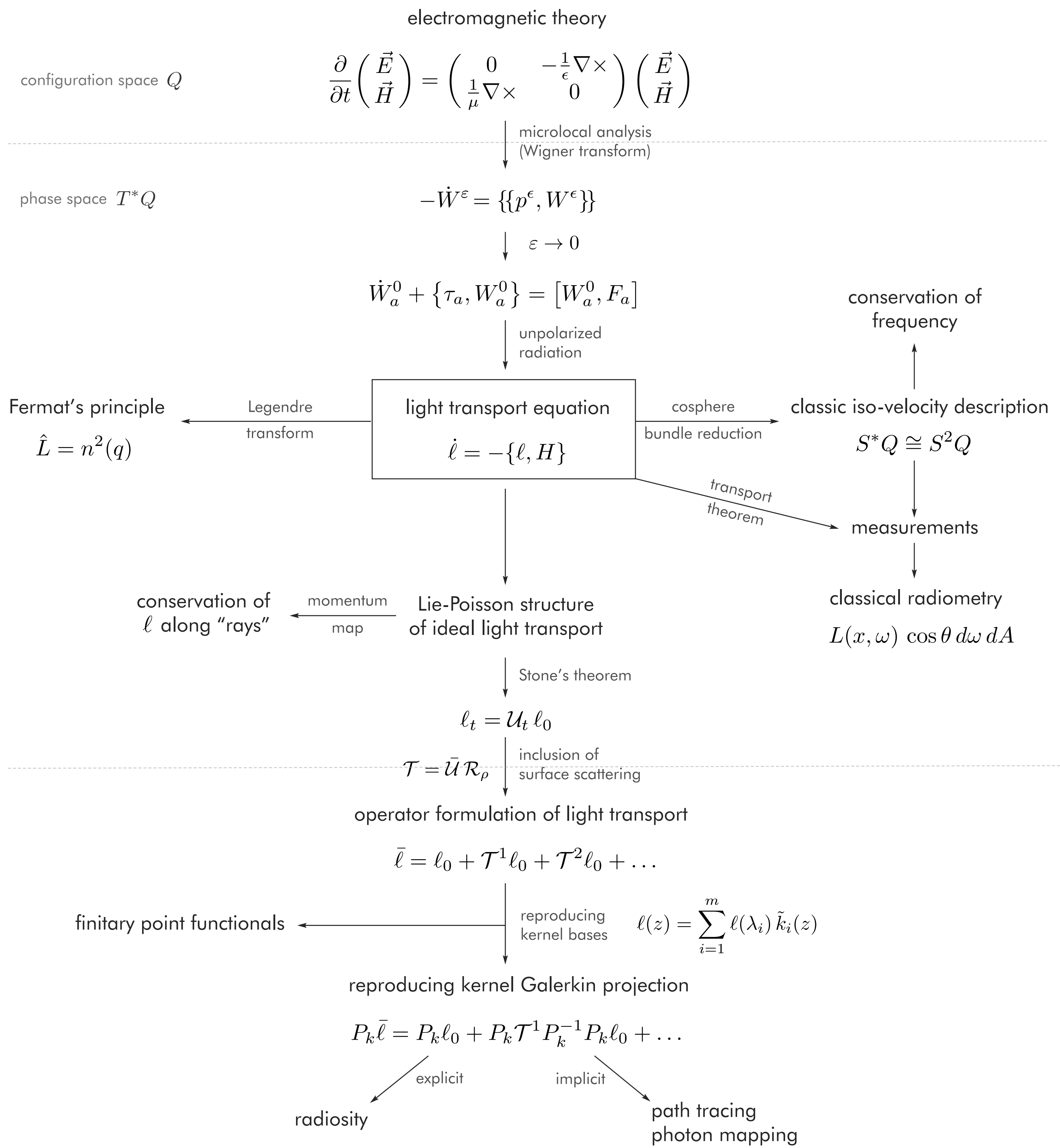
operator formulation of light transport

$$\bar{\ell} = \ell_0 + \mathcal{T}^1 \ell_0 + \mathcal{T}^2 \ell_0 + \dots$$


Stone's theorem
$\ell_t = \mathcal{U}_t \, \ell_0$
$\mathcal{T} = \bar{\mathcal{U}} \, \mathcal{R}_\rho$
inclusion of
surface scattering
operator formulation of light transport
$\bar{\ell} = \ell_0 + \mathcal{T}^1 \ell_0 + \mathcal{T}^2 \ell_0 + \ldots$
finitary point functionals
reproducing
kernel bases
$\ell(z) = \sum_{i=1}^{m} \ell(\lambda_i) \, \tilde{k}_i(z)$


Stone's theorem
$\ell_t = \mathcal{U}_t \, \ell_0$
$\mathcal{T} = \bar{\mathcal{U}} \, \mathcal{R}_\rho$
inclusion of surface scattering
operator formulation of light transport
$\bar{\ell} = \ell_0 + \mathcal{T}^1 \ell_0 + \mathcal{T}^2 \ell_0 + \ldots$
finitary point functionals
reproducing kernel bases
$\ell(z) = \sum_{i=1}^{m} \ell(\lambda_i) \, \tilde{k}_i(z)$
reproducing kernel Galerkin projection
$P_k \bar{\ell} = P_k \ell_0 + P_k \mathcal{T}^1 P_k^{-1} P_k \ell_0 + \ldots$

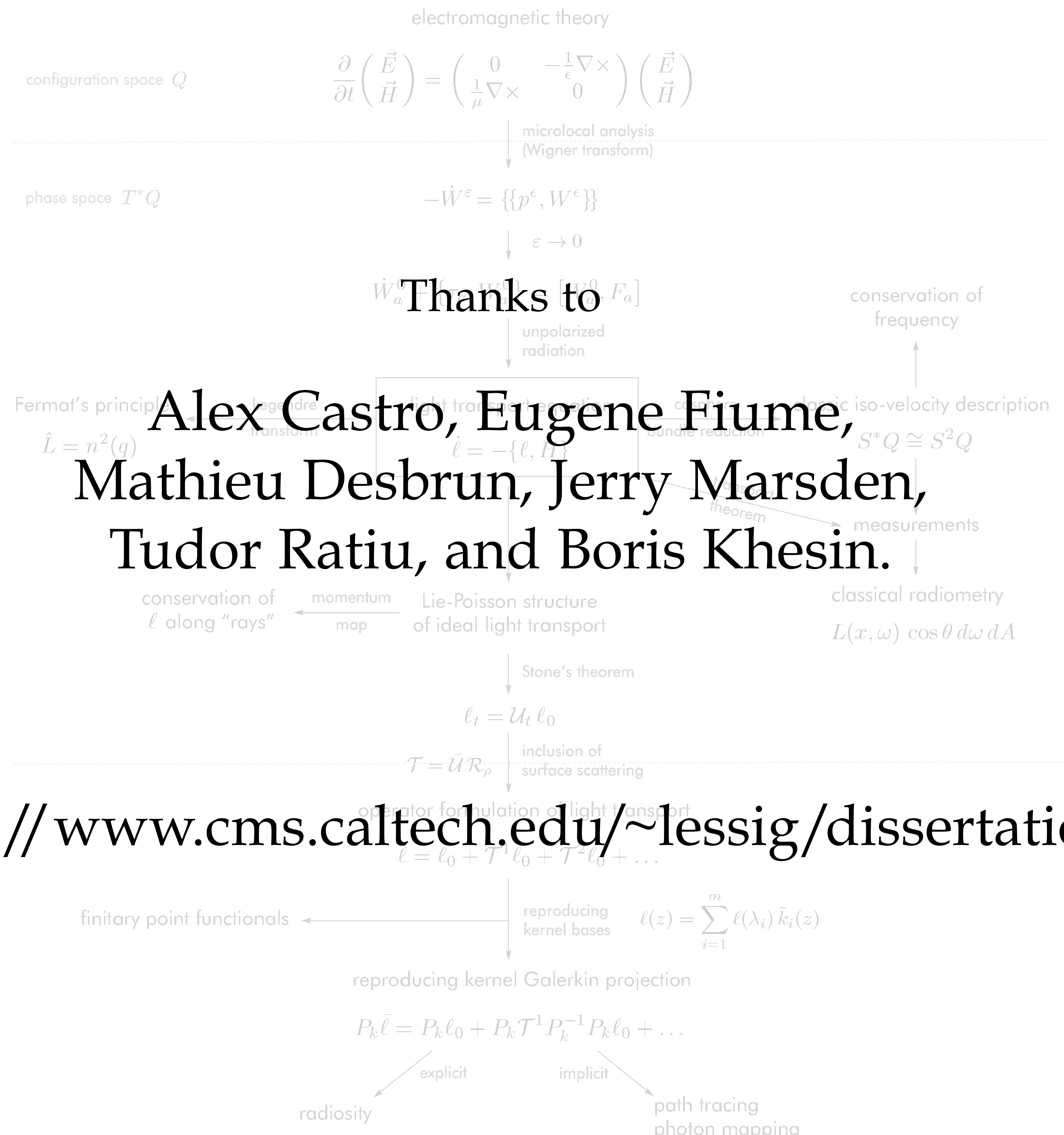

Stone's theorem
$\ell_t = \mathcal{U}_t \, \ell_0$
$\mathcal{T} = \bar{\mathcal{U}} \, \mathcal{R}_\rho$
inclusion of
surface scattering
operator formulation of light transport
$\bar{\ell} = \ell_0 + \mathcal{T}^1 \ell_0 + \mathcal{T}^2 \ell_0 + \dots$
finitary point functionals
reproducing
kernel bases
$\ell(z) = \sum_{i=1}^{m} \ell(\lambda_i) \, \tilde{k}_i(z)$
reproducing kernel Galerkin projection
$P_k \bar{\ell} = P_k \ell_0 + P_k \mathcal{T}^1 P_k^{-1} P_k \ell_0 + \dots$
explicit
implicit
radiosity
path tracing
photon mapping



electromagnetic theory
configuration space Q
$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$
microlocal analysis
(Wigner transform)
phase space T^*Q
$-\dot{W}^{\varepsilon} = \{\{p^{\epsilon}, W^{\epsilon}\}\}$
$\varepsilon \to 0$
$\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]$
unpolarized
radiation
conservation of
frequency
Fermat's principle
$\hat{L} = n^2(q)$
Legendre
transform
light transport equation
$\dot{\ell} = -\{\ell, H\}$
cosphere
bundle reduction
classic iso-velocity description
$S^*Q \cong S^2Q$
transport
theorem
measurements
classical radiometry
$L(x,\omega)\,\cos\theta\,d\omega\,dA$
conservation of
ℓ along "rays"
momentum
map
Lie-Poisson structure
of ideal light transport
Stone's theorem
$\ell_t = \mathcal{U}_t\,\ell_0$
$\mathcal{T} = \bar{\mathcal{U}}\,\mathcal{R}_{\rho}$
inclusion of
surface scattering
operator formulation of light transport
$\bar{\ell} = \ell_0 + \mathcal{T}^1\ell_0 + \mathcal{T}^2\ell_0 + \dots$
finitary point functionals
reproducing
kernel bases
$\ell(z) = \sum_{i=1}^{m} \ell(\lambda_i)\,\tilde{k}_i(z)$
reproducing kernel Galerkin projection
$P_k\bar{\ell} = P_k\ell_0 + P_k\mathcal{T}^1 P_k^{-1} P_k\ell_0 + \dots$
explicit
implicit
radiosity
path tracing
photon mapping

What next?

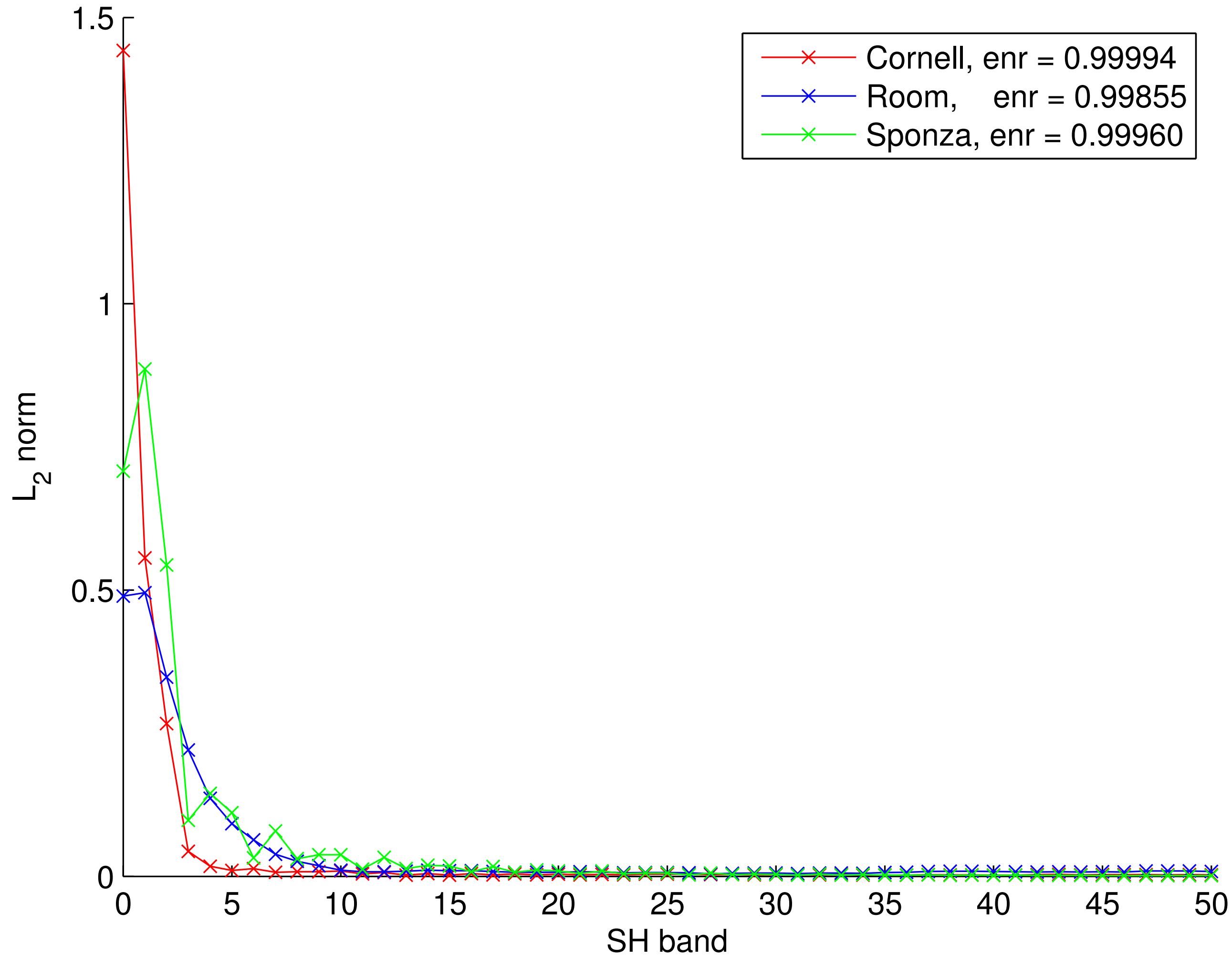
- Evolution operator.
- Derivative ray tracing.
- Characterization of steady state solutions.
- Function spaces of light transport.
- ...



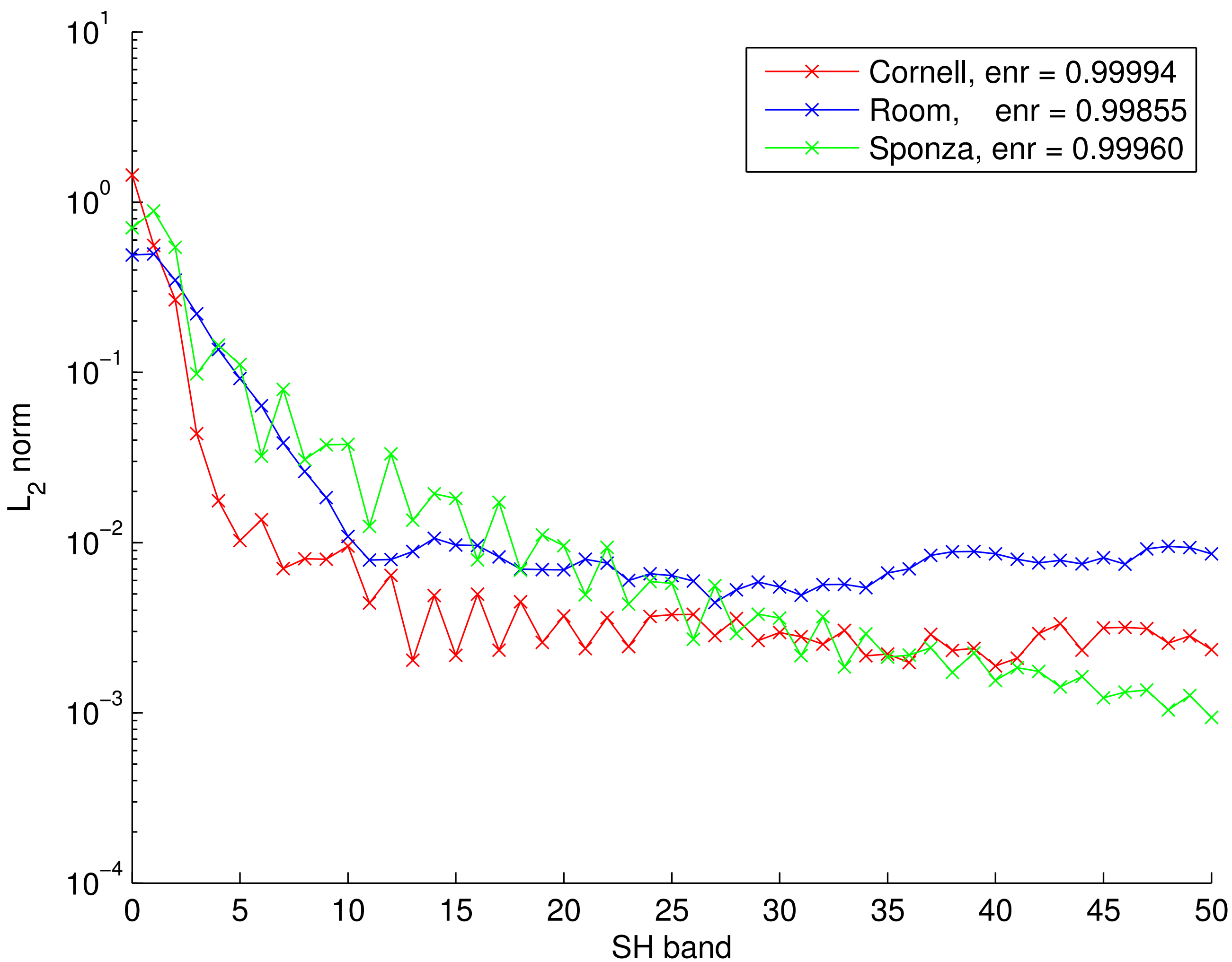
Thanks to

Alex Castro, Eugene Fiume,
Mathieu Desbrun, Jerry Marsden,
Tudor Ratiu, and Boris Khesin.

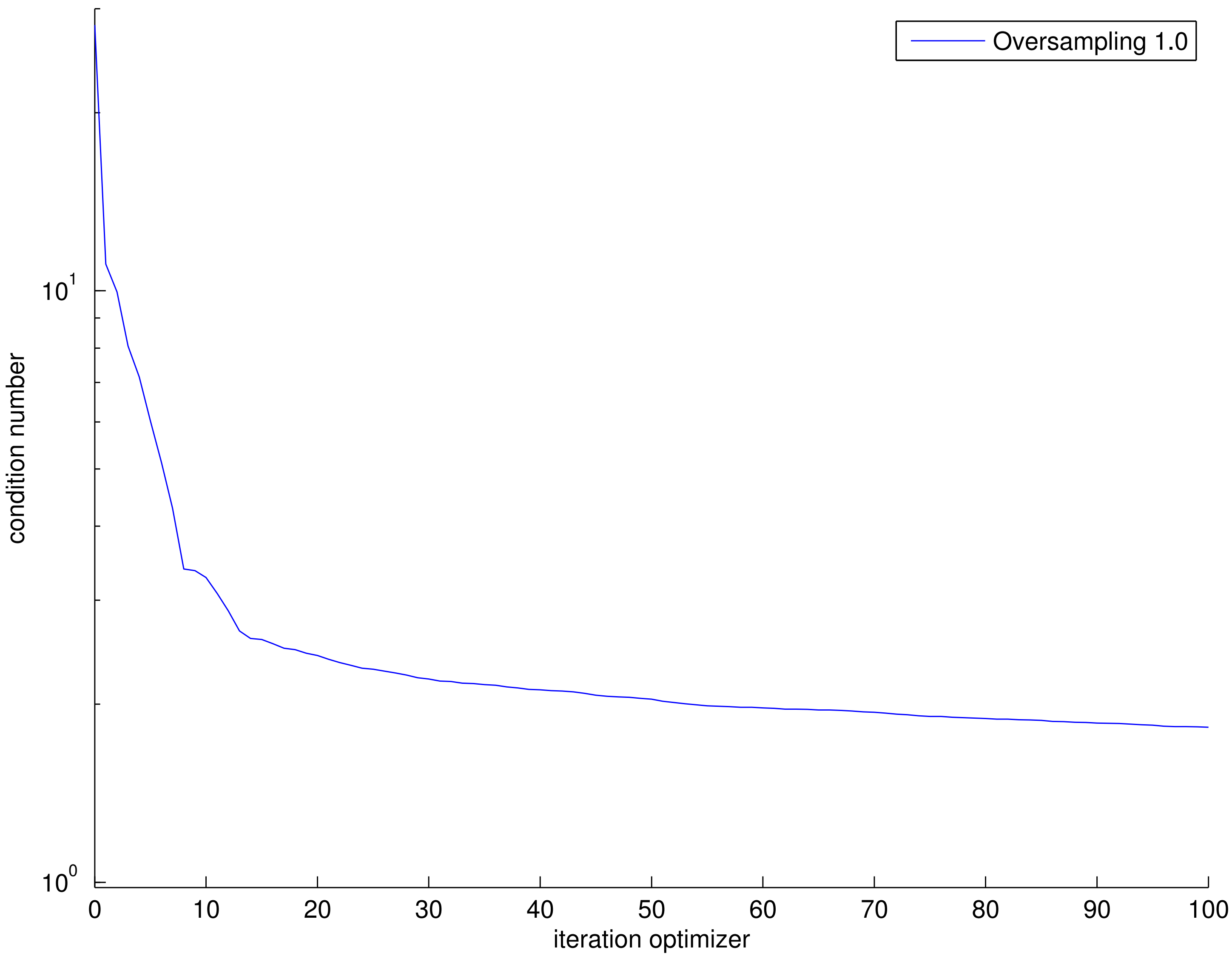
http://www.cms.caltech.edu/~lessig/dissertation/

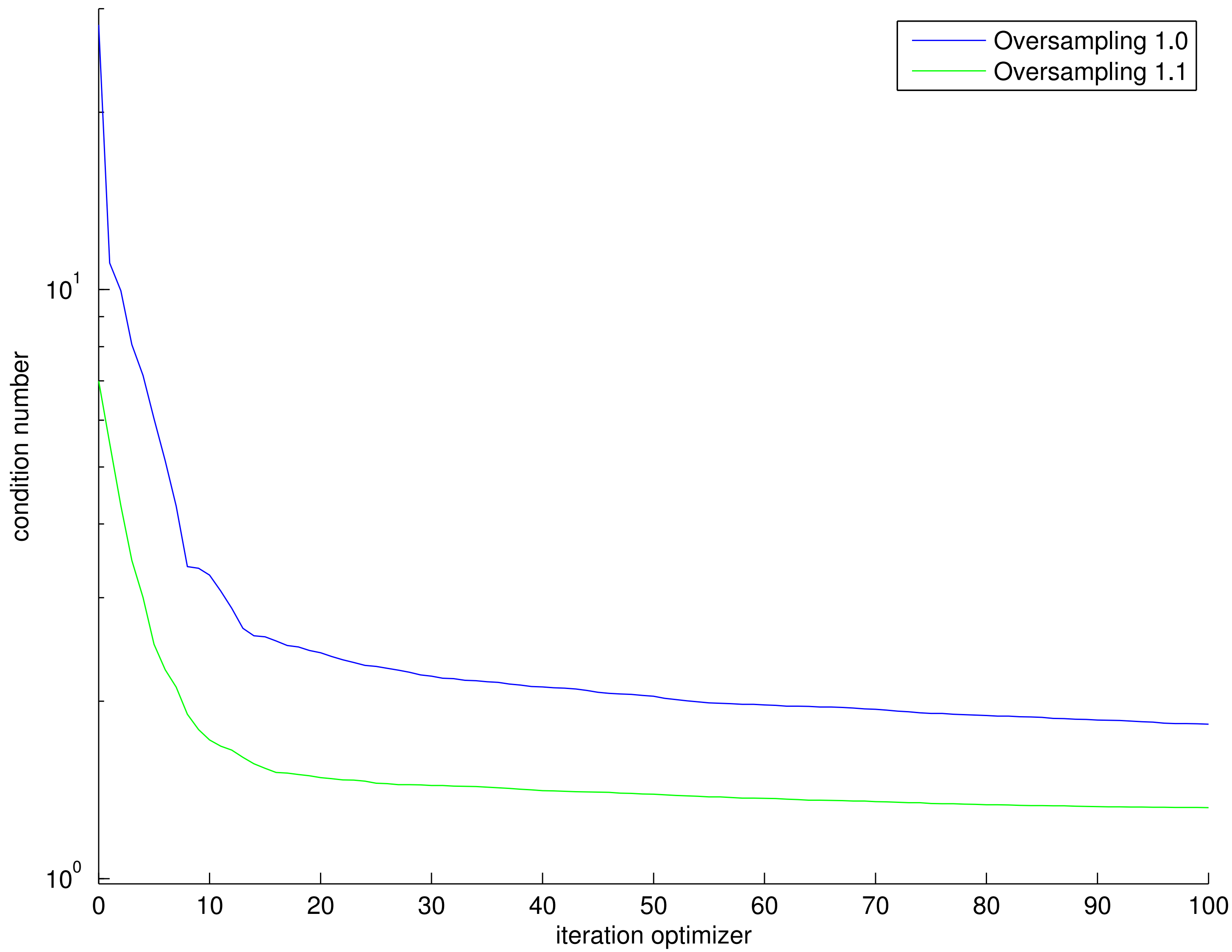


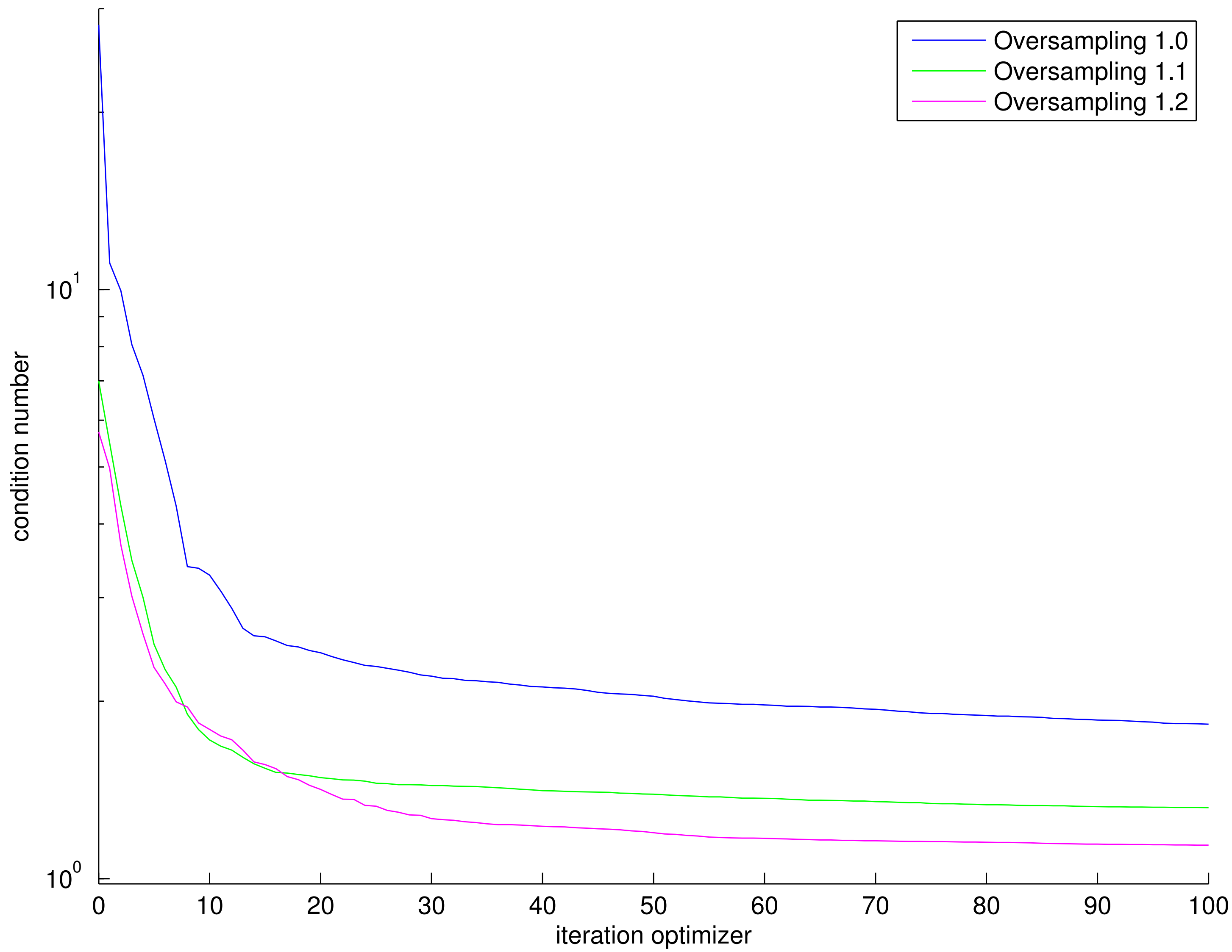
Cornell, enr = 0.99994
Room, enr = 0.99855
Sponza, enr = 0.99960
L2 norm
SH band
0
0.5
1
1.5
0
5
10
15
20
25
30
35
40
45
50

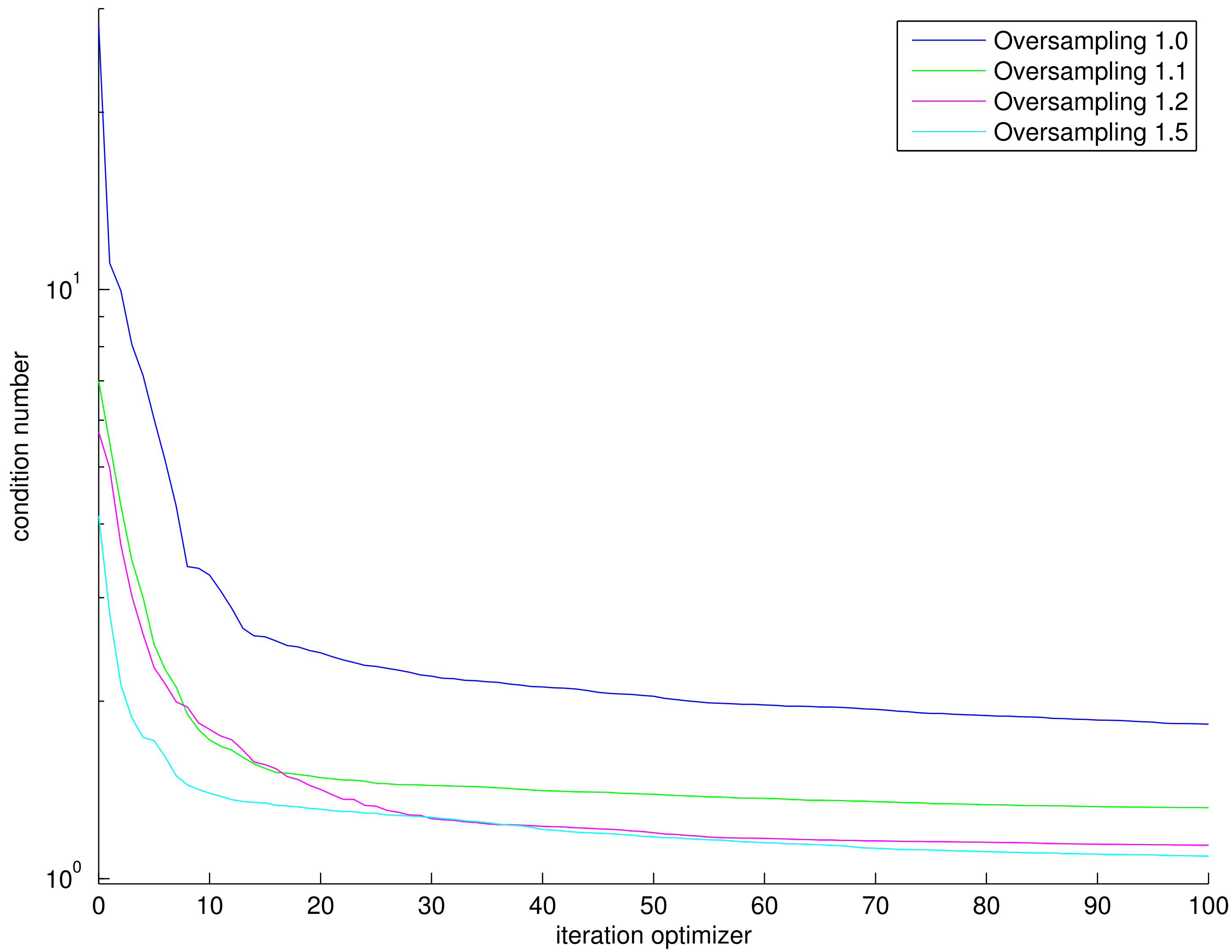


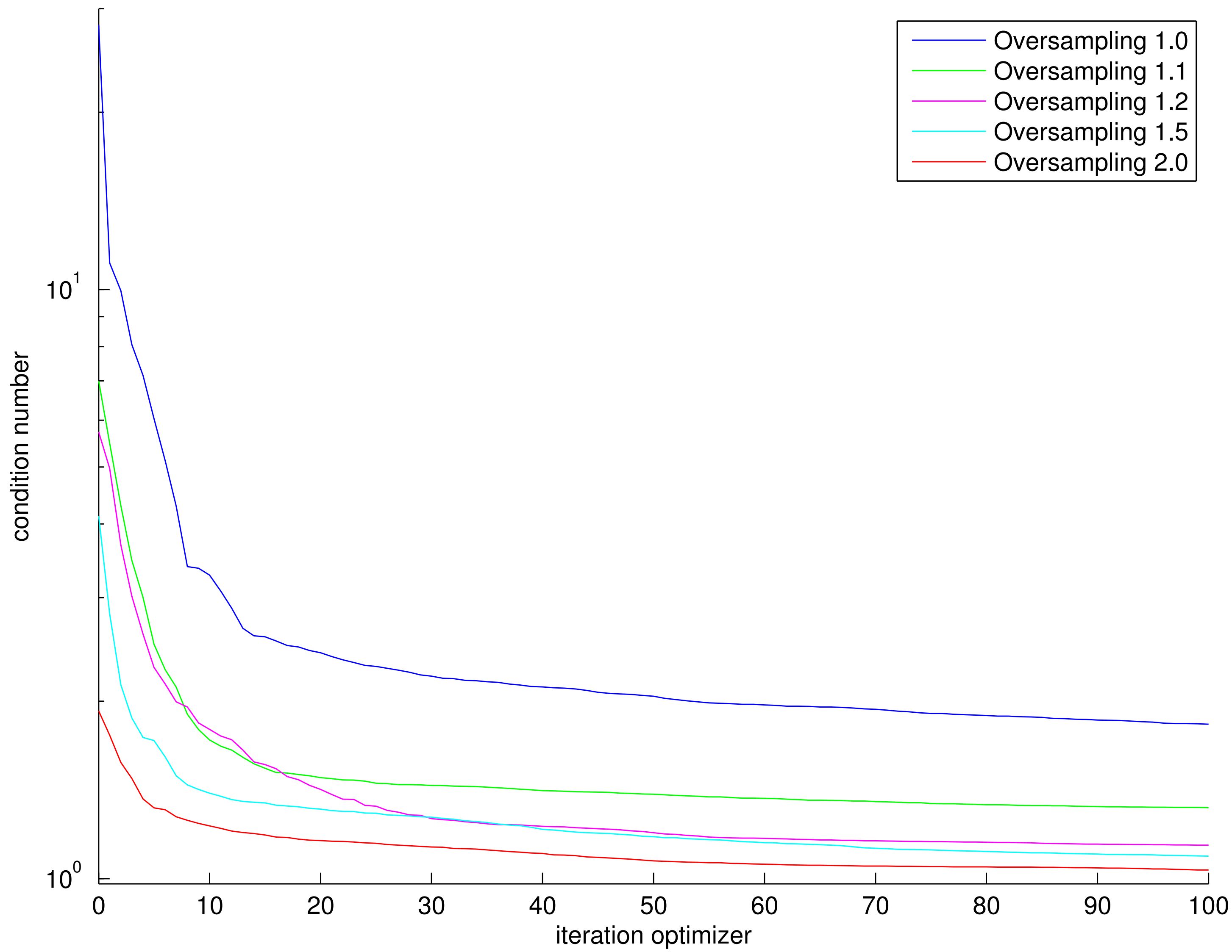
Cornell, enr = 0.99994
Room, enr = 0.99855
Sponza, enr = 0.99960
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
L_2 norm
0
5
10
15
20
25
30
35
40
45
50
SH band


Oversampling 1.0
condition number
10^1
10^0
0
10
20
30
40
50
60
70
80
90
100
iteration optimizer

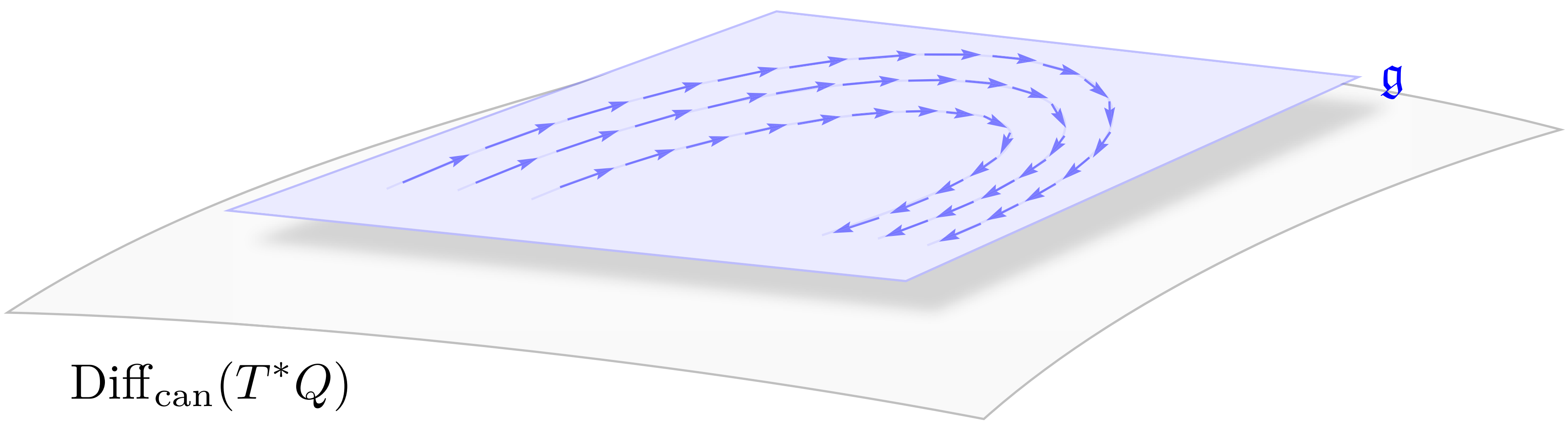

Oversampling 1.0
Oversampling 1.1
condition number
10^1
10^0
0
10
20
30
40
50
60
70
80
90
100
iteration optimizer

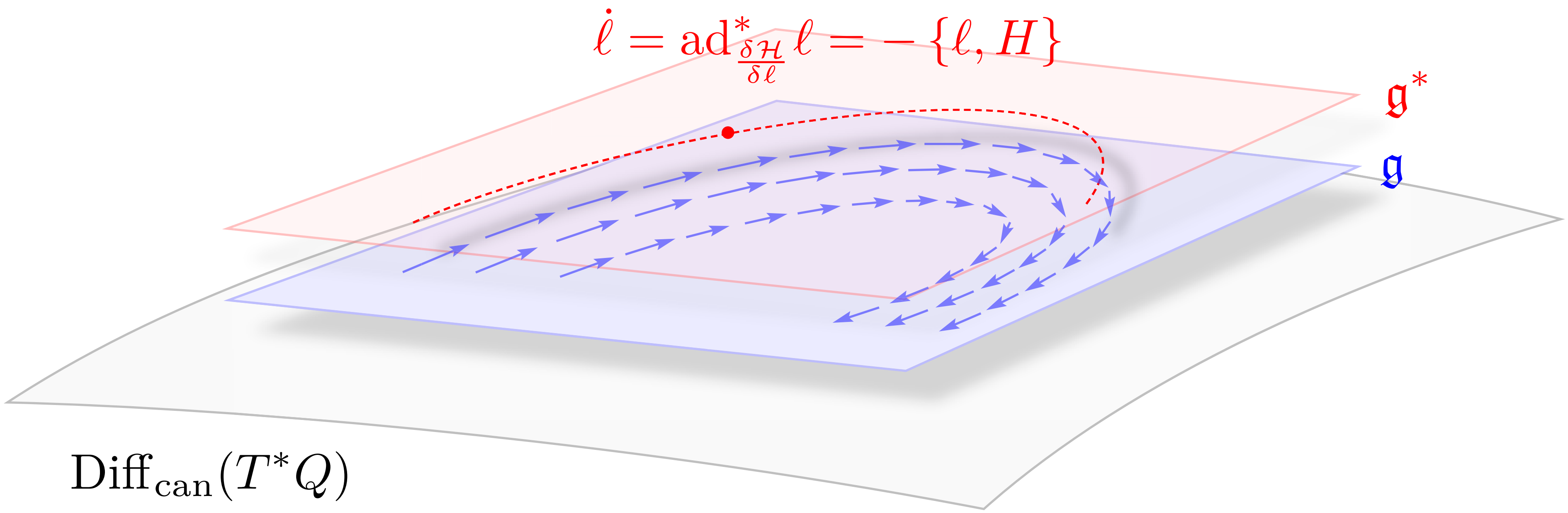

Oversampling 1.0
Oversampling 1.1
Oversampling 1.2
condition number
iteration optimizer
10^1
10^0
0
10
20
30
40
50
60
70
80
90
100

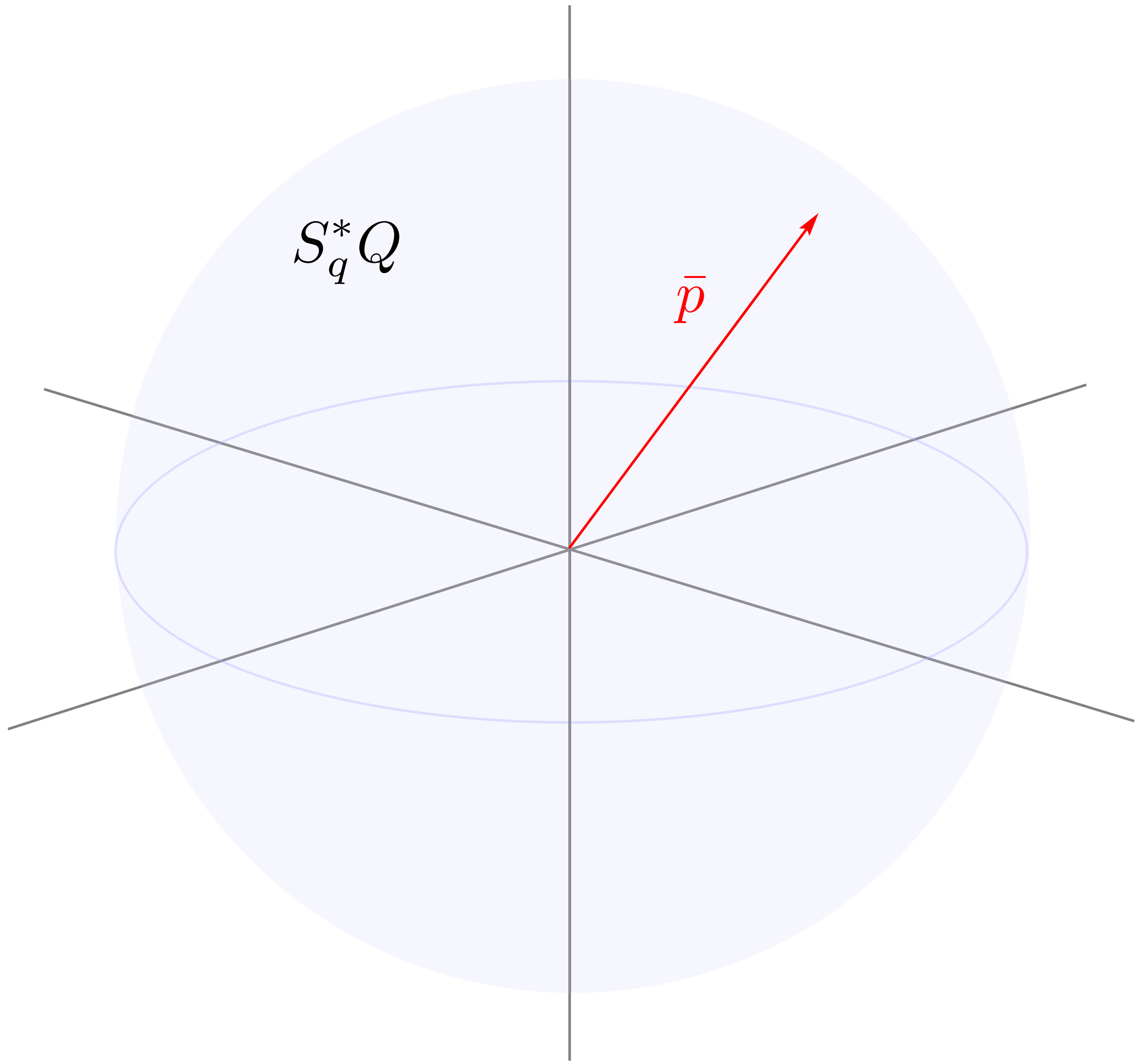

Oversampling 1.0
Oversampling 1.1
Oversampling 1.2
Oversampling 1.5
condition number
10^1
10^0
0
10
20
30
40
50
60
70
80
90
100
iteration optimizer

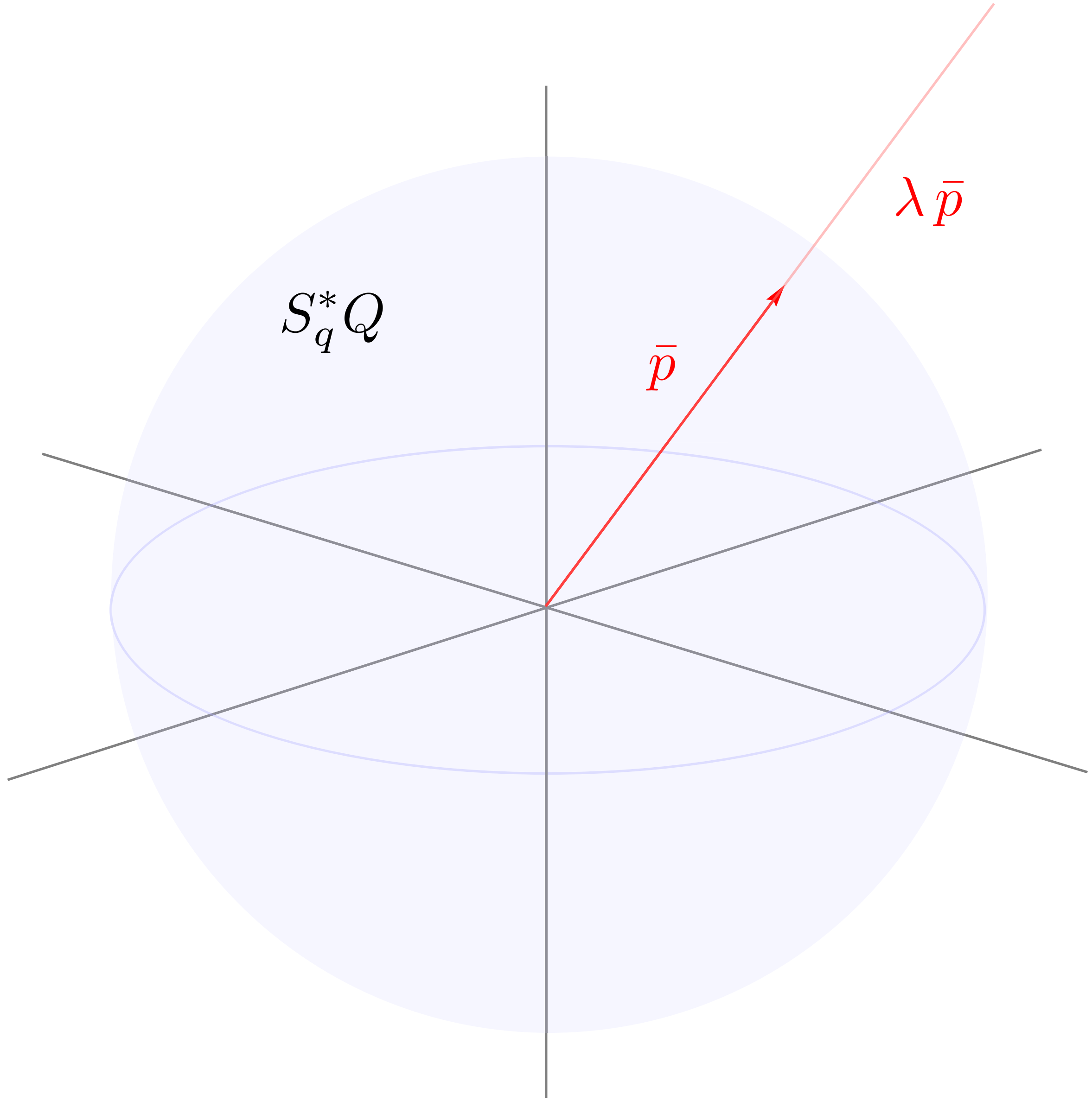


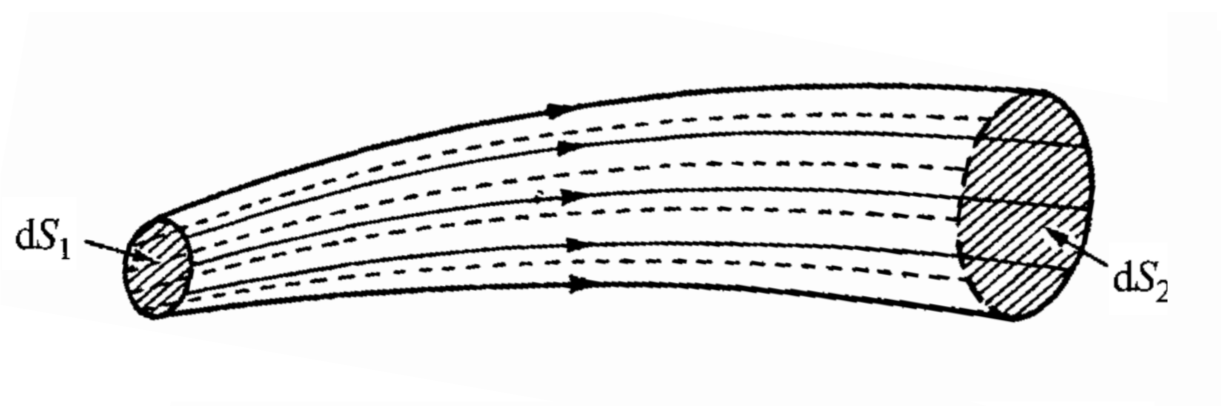
Oversampling 1.0
Oversampling 1.1
Oversampling 1.2
Oversampling 1.5
Oversampling 2.0
condition number
10^1
10^0
0
10
20
30
40
50
60
70
80
90
100
iteration optimizer

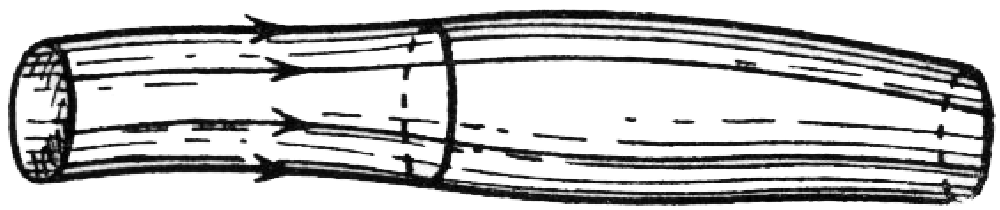

$\mathfrak{g}$
$\mathrm{Diff}_{\mathrm{can}}(T^*Q)$

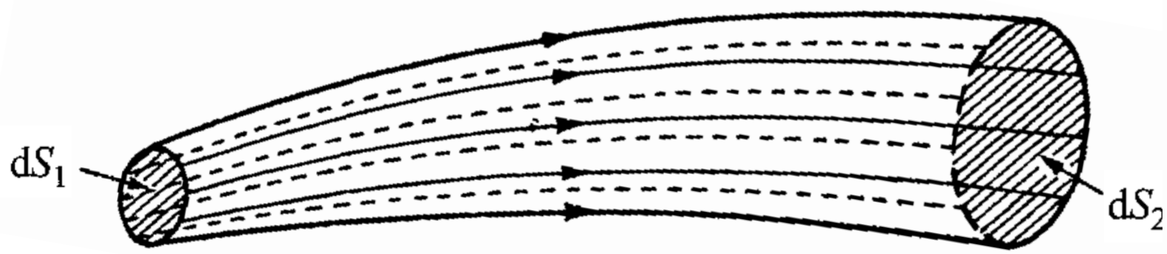

$\dot{\ell} = \mathrm{ad}^*_{\frac{\delta \mathcal{H}}{\delta \ell}} \ell = -\{\ell, H\}$
$\mathfrak{g}^*$
$\mathfrak{g}$
$\mathrm{Diff}_{\mathrm{can}}(T^*Q)$

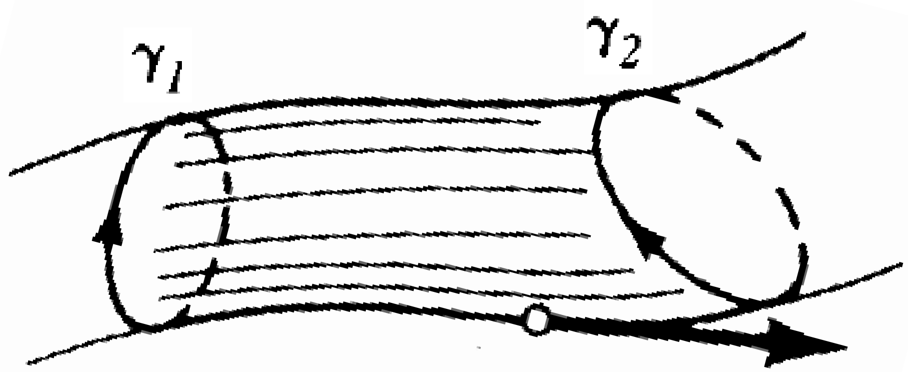

S_q^*Q
$\bar{p}$

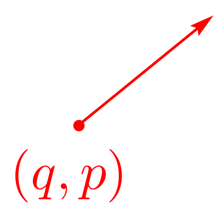

$S_q^* Q$
$\bar{p}$
$\lambda\,\bar{p}$

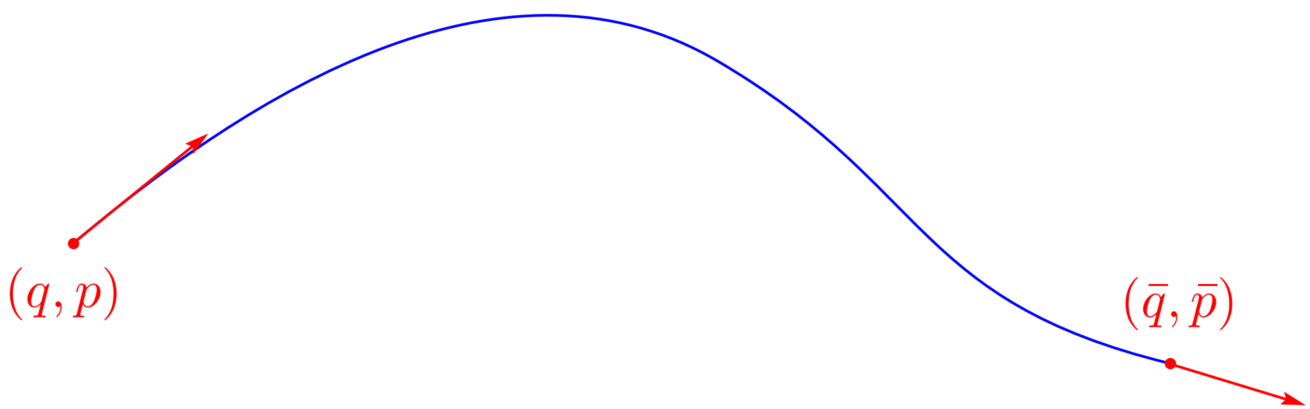

dS_1
dS_2




dS_1
dS_2


γ_1
γ_2


(q, p)



(q, p)
$(\bar{q}, \bar{p})$