The Geometry of Light Transport

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quantum electrodynamics

quantum electrodynamics

↓ large number of photons

Maxwell's equations

quantum electrodynamics

↓ large number of photons

Maxwell's equations

↓ short wavelength limit
neglect of polarization

geometric optics

quantum electrodynamics

↓ large number of photons

Maxwell's equations

↓ short wavelength limit
neglect of polarization

light transport theory

quantum electrodynamics

↓ large number of photons

Maxwell's equations

↓ short wavelength limit
neglect of polarization

light transport theory

↓ neglect of intensity

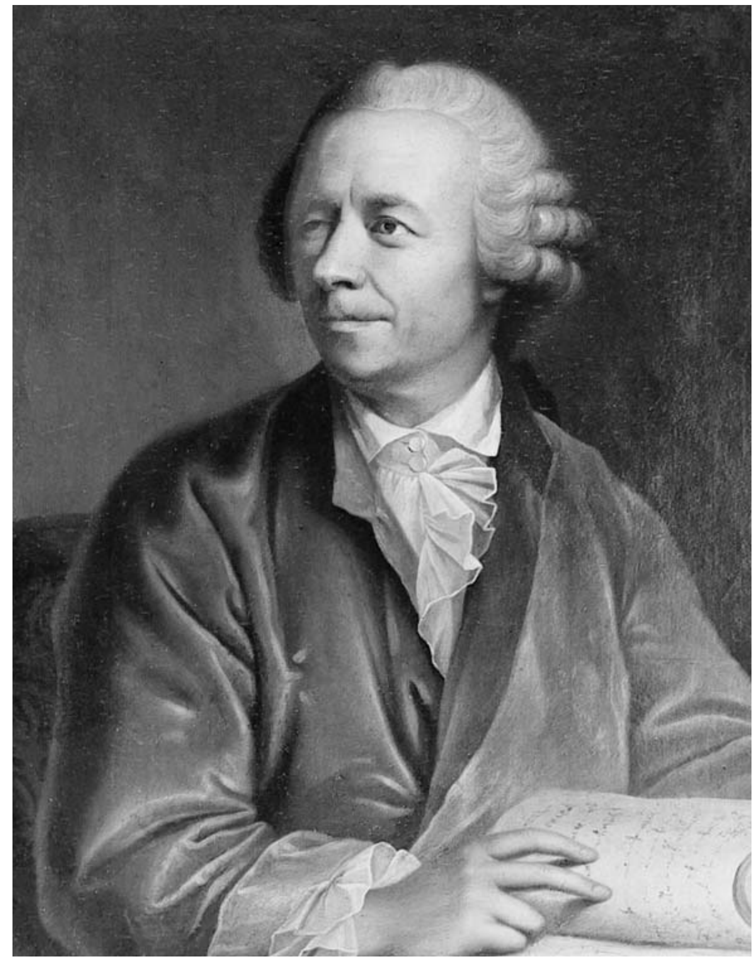
geometric optics

http://www.kuttaka.org/~JHL/LambertBild2.jpg

Johann Heinrich Lambert

http://www.kuttaka.org/~JHL/LambertBild2.jpg

Johann Heinrich Lambert



http://www.ega-math.narod.ru/Bell/IMG/Euler1.jpg

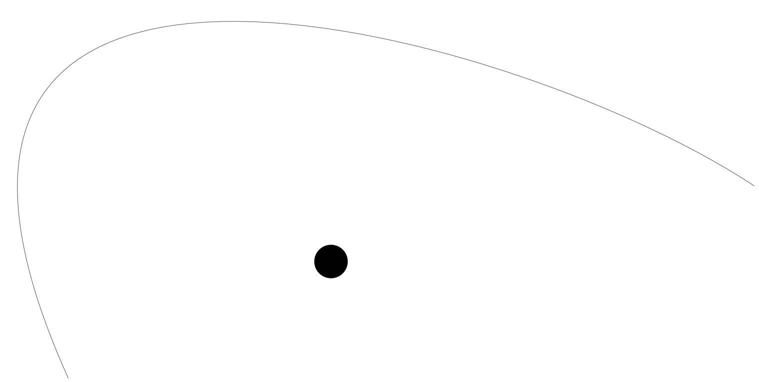
Leonhard Euler

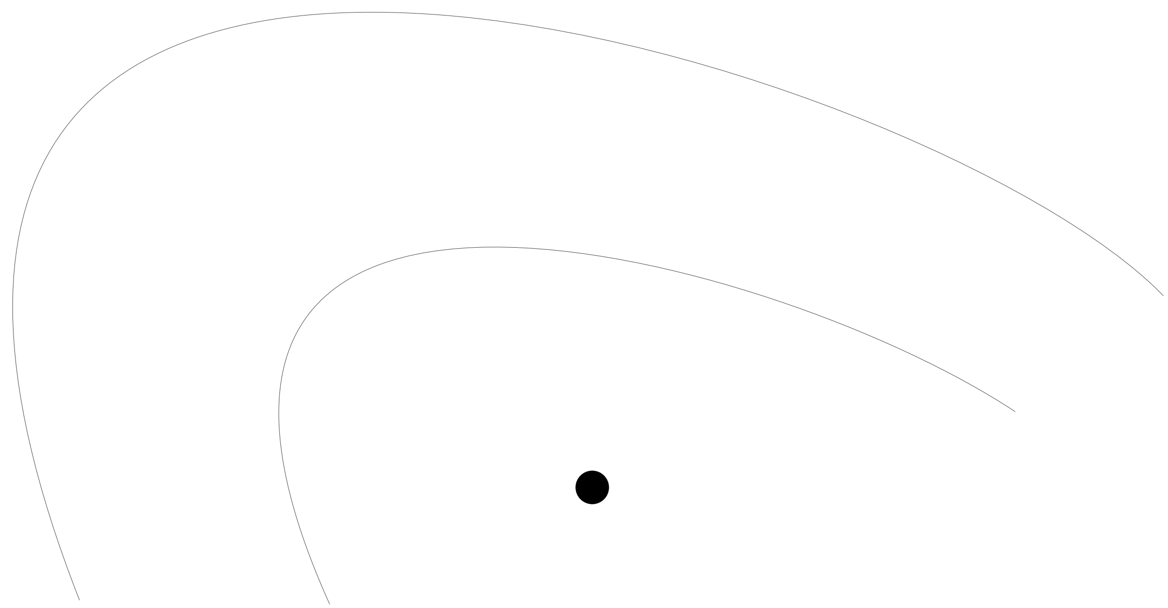
"Theoretical photometry constitutes a case of 'arrested development', and has remained basically unchanged since 1760 while the rest of physics has swept triumphantly ahead. In recent years, however, the increasing needs [. . .] have made the absurdly antiquated concepts of traditional photometric theory more and more untenable."[1]

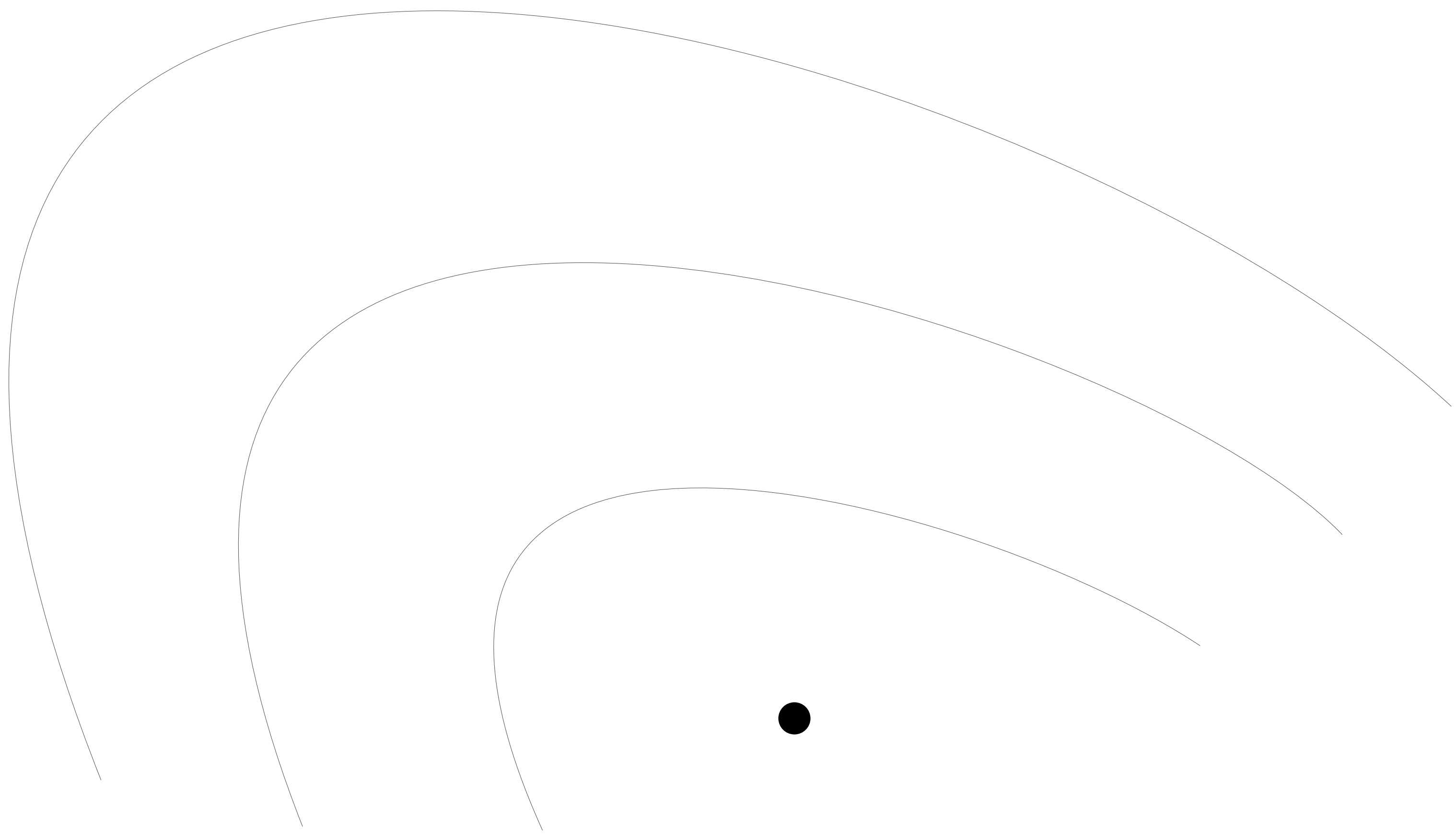
1. Gershun, A. *The Light Field*, Translated by P. Moon, G. Timoshenko, Originally published in Russian (Moscow 1936). Journal of Mathematics and Physics 18 (1939): 51-151, from the translators preface.

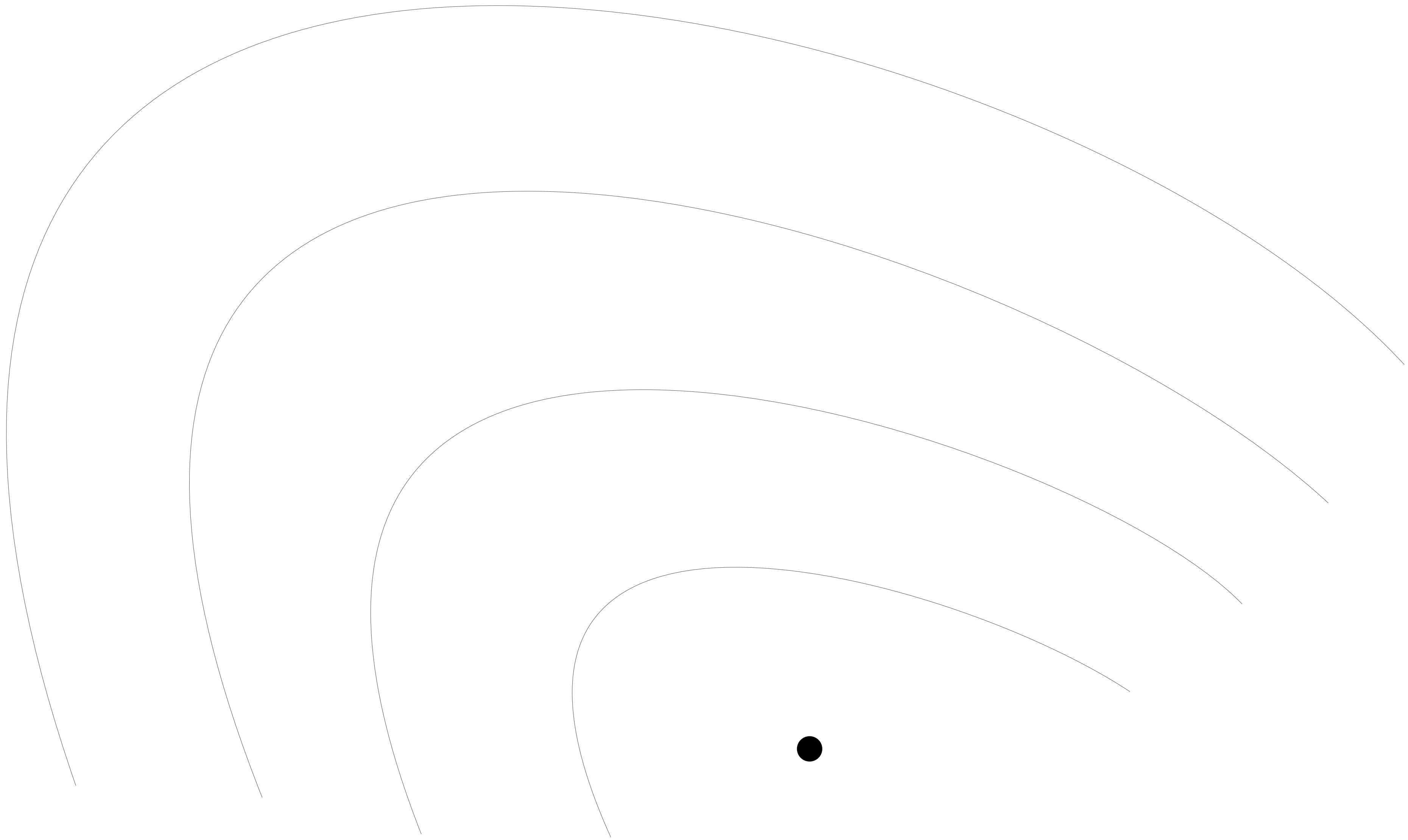
configuration space $Q \subset \mathbb{R}^3$

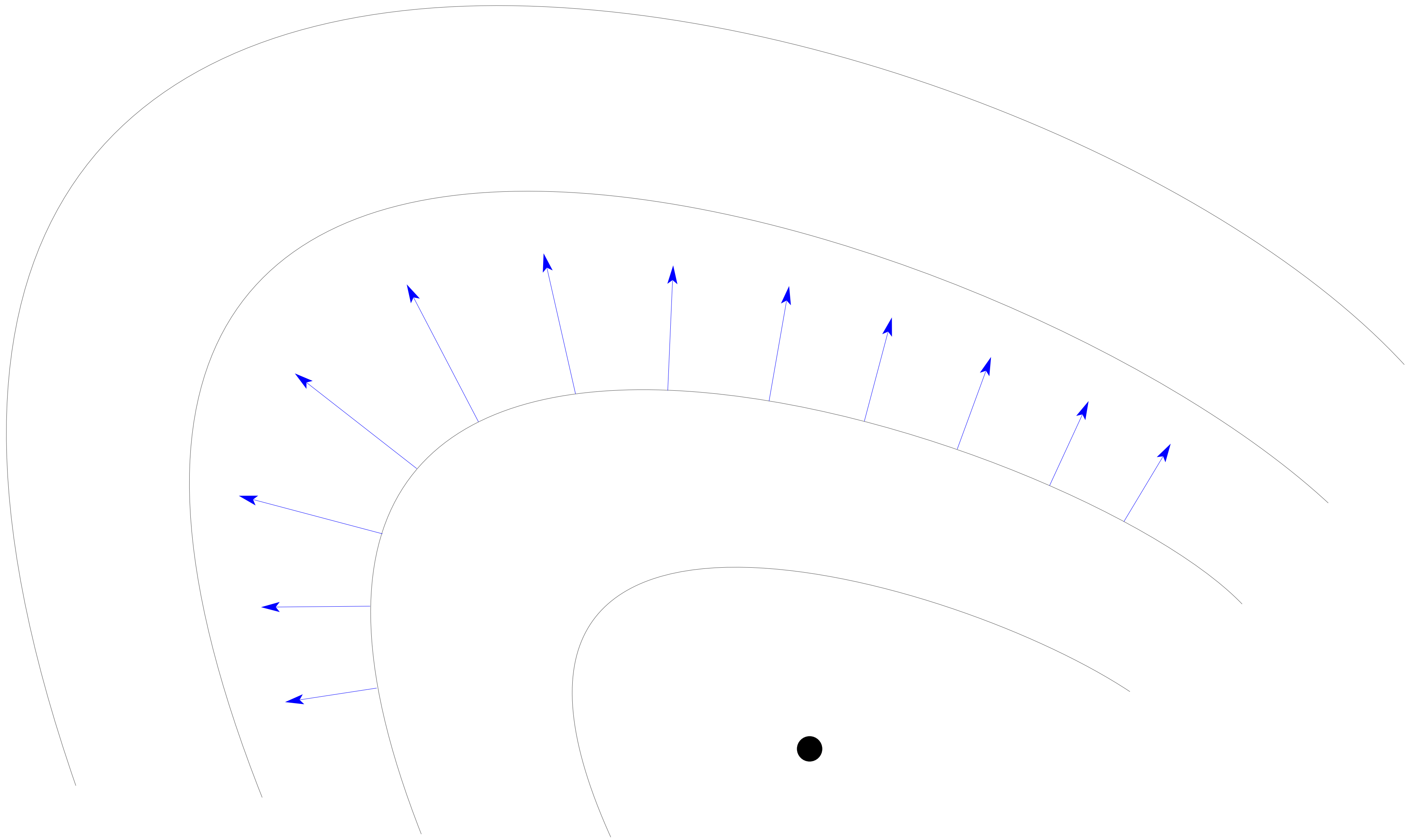
$$\frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

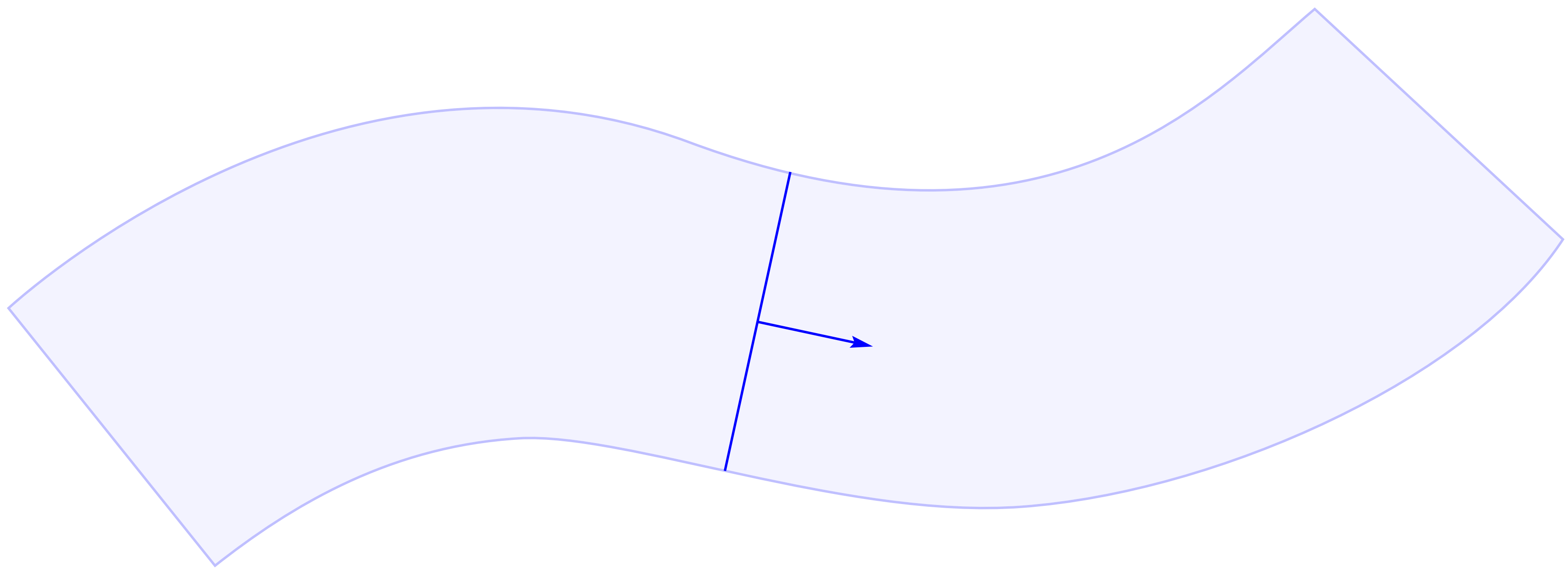


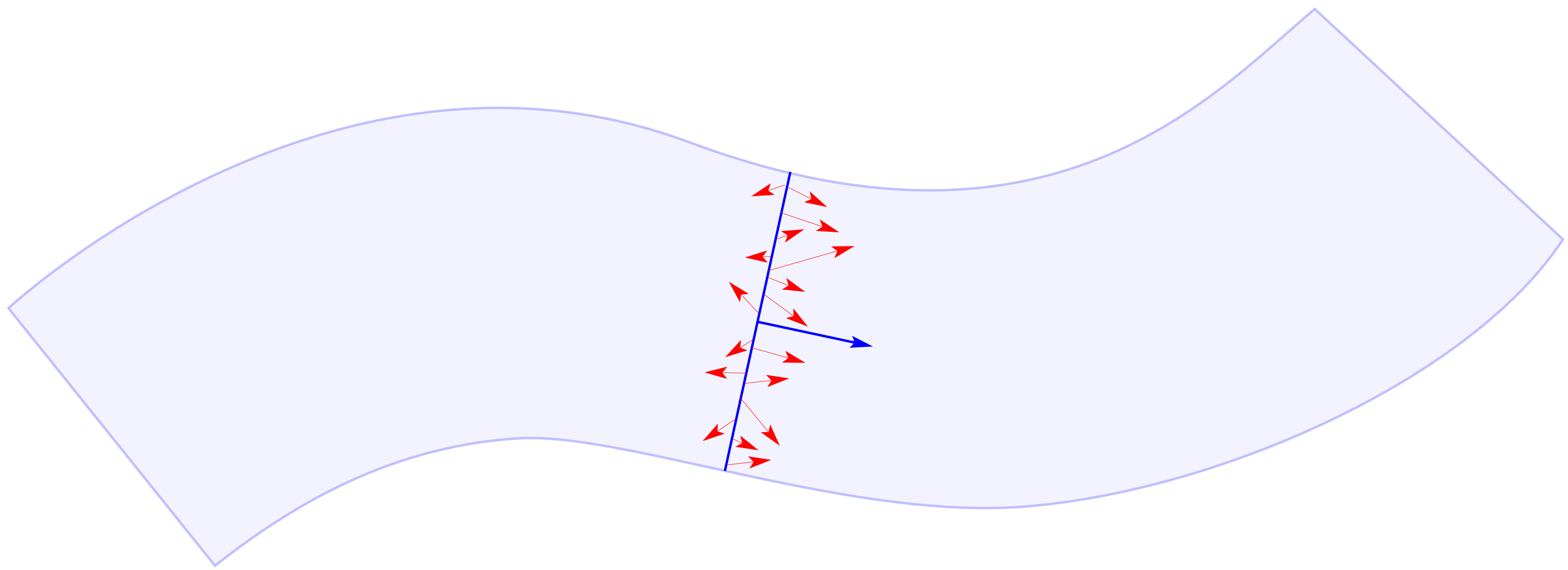


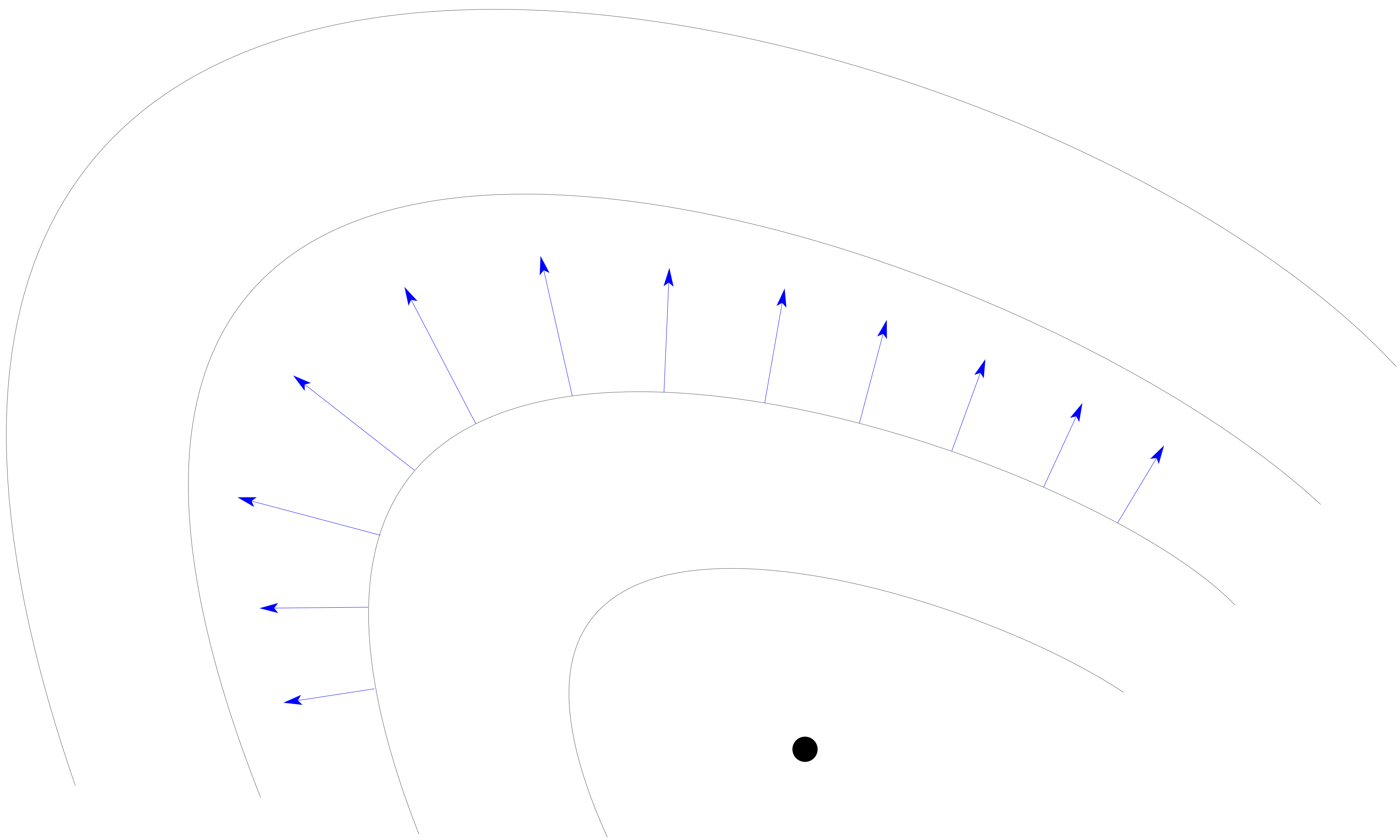


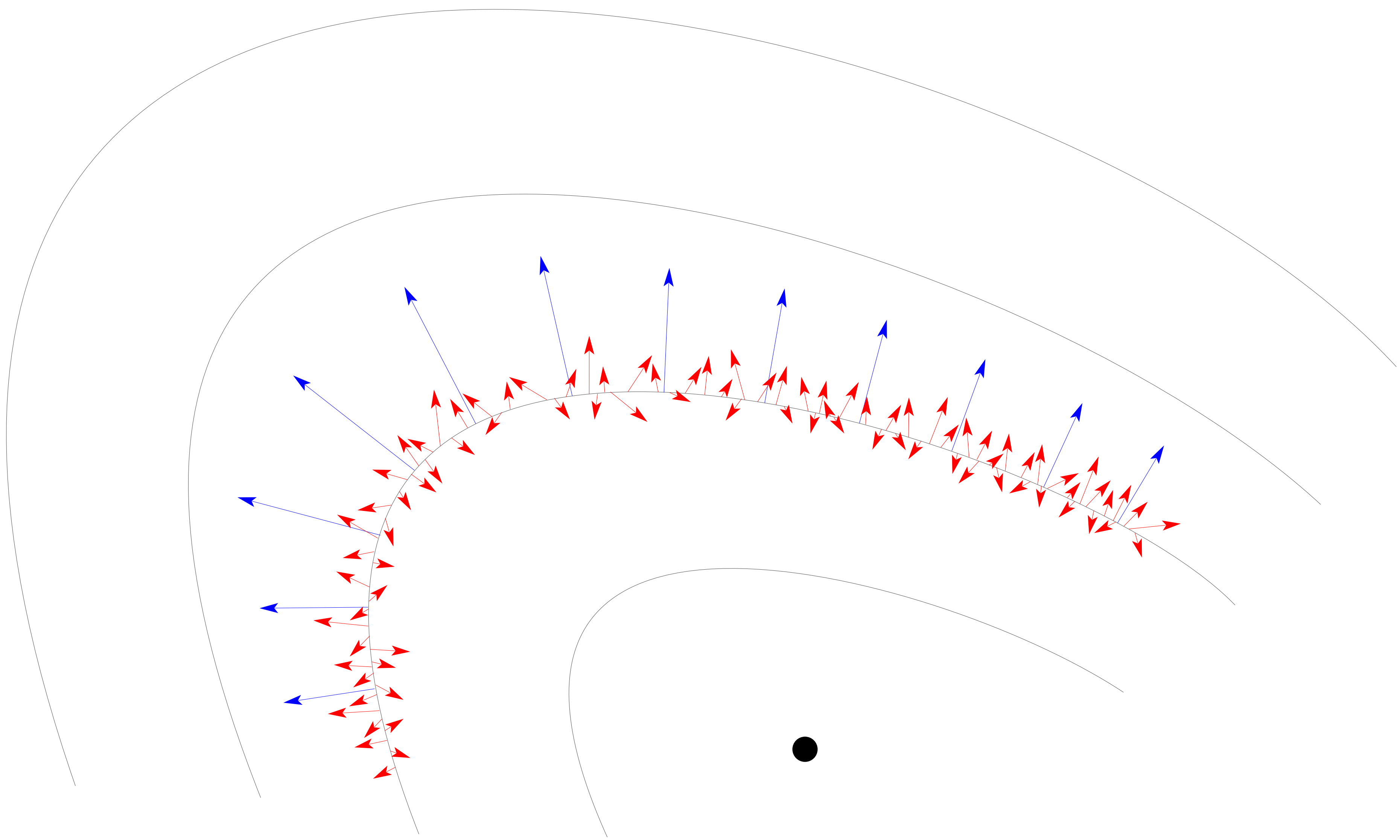


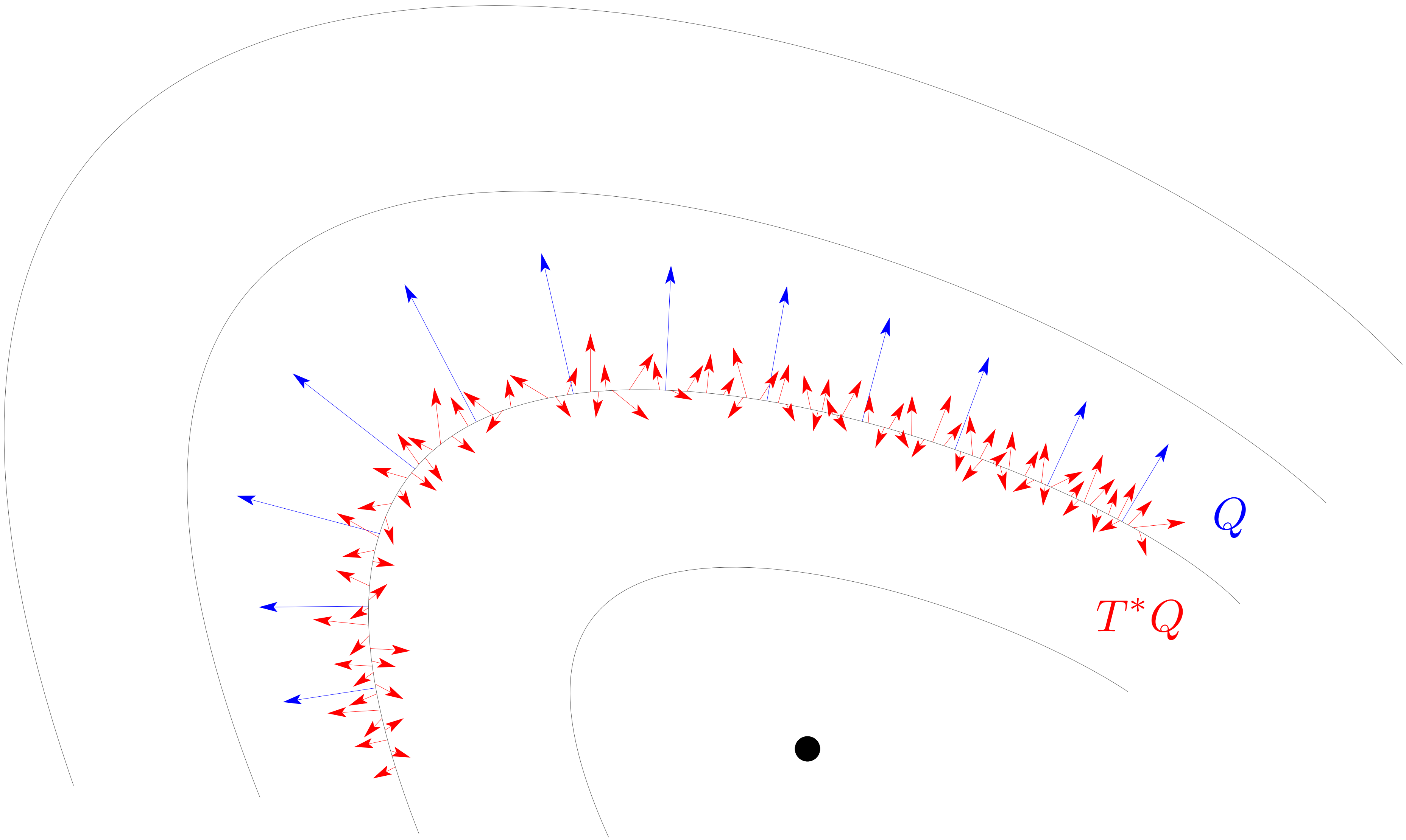












Q
T^*Q

configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

↓ microlocal analysis (Wigner transform)

phase space T^*Q

$$\dot{W}^\varepsilon = -\{\{p^\varepsilon, W^\varepsilon\}\}$$

configuration space $Q \subset \mathbb{R}^3$

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microlocal analysis
(Wigner transform)

phase space T^*Q

$$-\dot{W}^{\varepsilon} = \frac{1}{\varepsilon}\,[p^{\varepsilon}, W^{\varepsilon}] + \frac{1}{i}\{p^{\varepsilon}, W^{\varepsilon}\} + O(\varepsilon)$$

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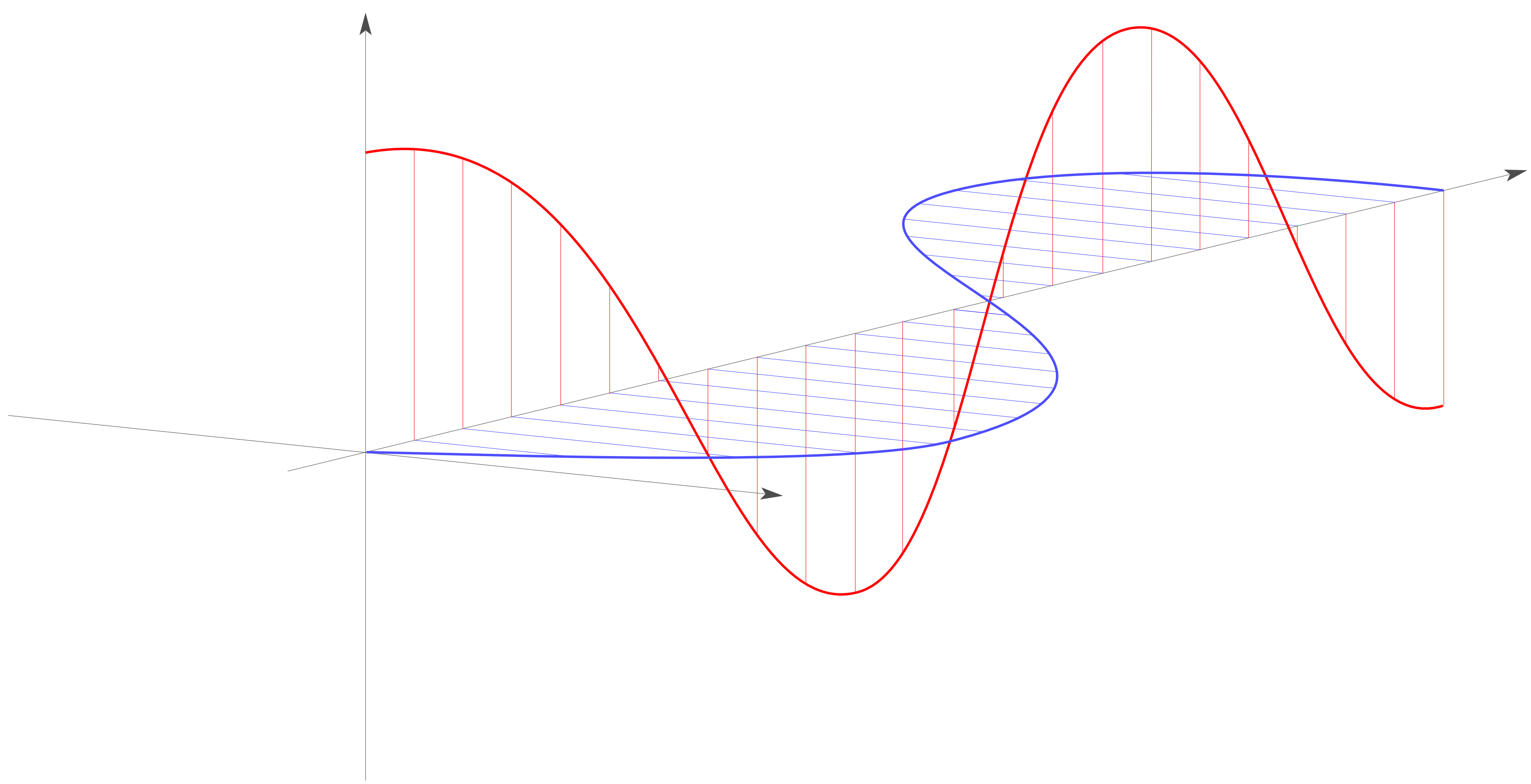

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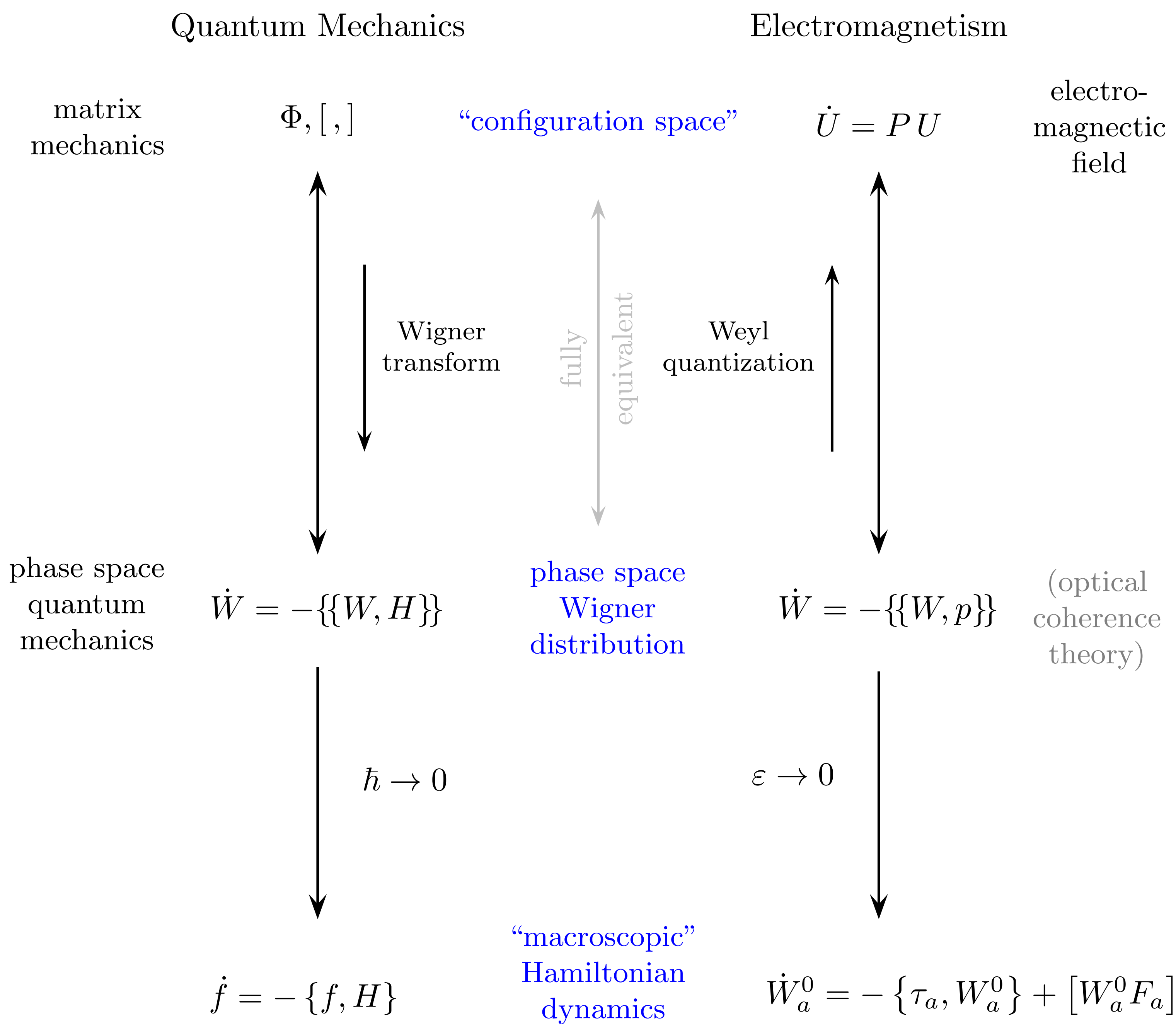
↓ microlocal analysis (Wigner transform)

phase space T^*Q

$$-\dot{W}^{\varepsilon} = \frac{1}{\varepsilon}\,[p^{\varepsilon}, W^{\varepsilon}] + \frac{1}{i}\{p^{\varepsilon}, W^{\varepsilon}\} + O(\varepsilon)$$

↓ $\varepsilon \to 0$

$$\dot{W}_a^0 + \left\{\tau_a, W_a^0\right\} = \left[W_a^0, F_a\right]$$


Quantum Mechanics
Electromagnetism
matrix mechanics
$\Phi, [\,,\,]$
“configuration space”
$\dot{U} = P\,U$
electro-magnectic field
Wigner transform
fully equivalent
Weyl quantization
phase space quantum mechanics
$\dot{W} = -\{\{W, H\}\}$
phase space Wigner distribution
$\dot{W} = -\{\{W, p\}\}$
(optical coherence theory)
$\hbar \to 0$
$\varepsilon \to 0$
$\dot{f} = -\{f, H\}$
“macroscopic” Hamiltonian dynamics
$\dot{W}^0_a = -\left\{\tau_a, W^0_a\right\} + \left[W^0_a F_a\right]$

configuration space $Q \subset \mathbb{R}^3$

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$\downarrow$ microlocal analysis (Wigner transform)

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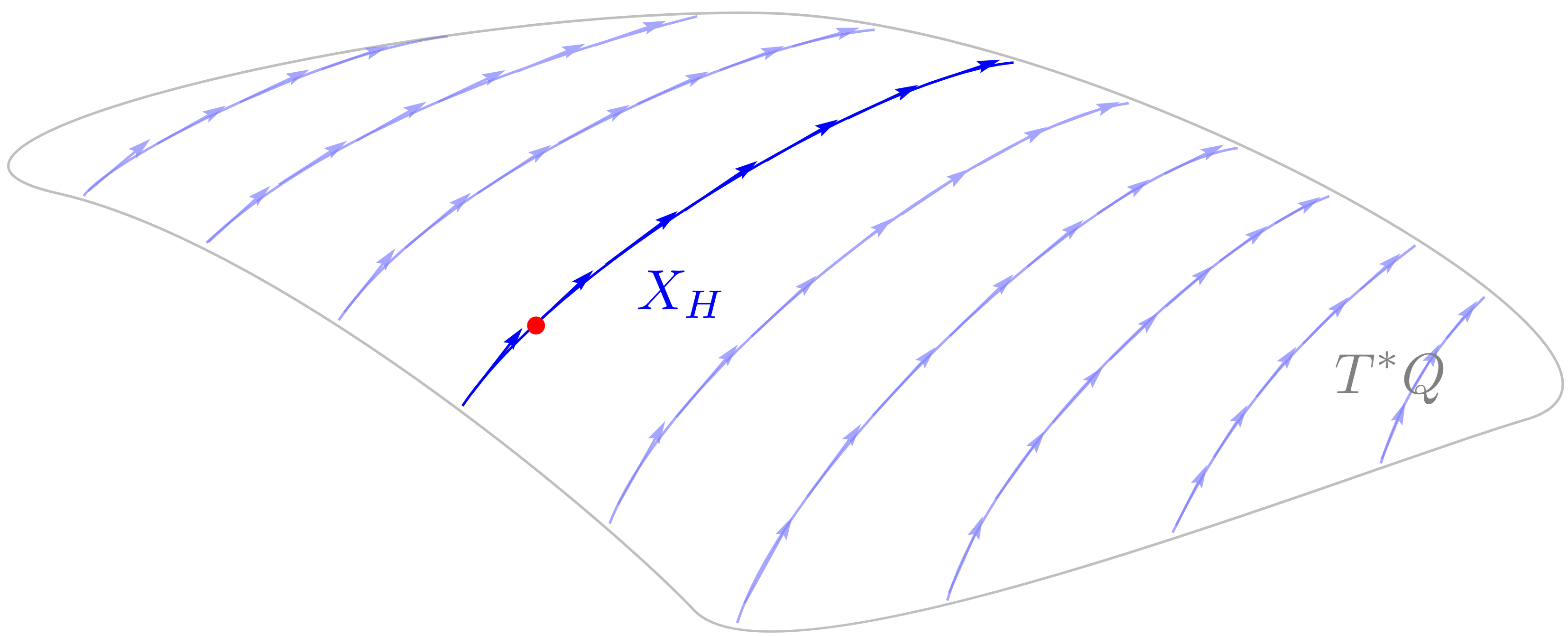
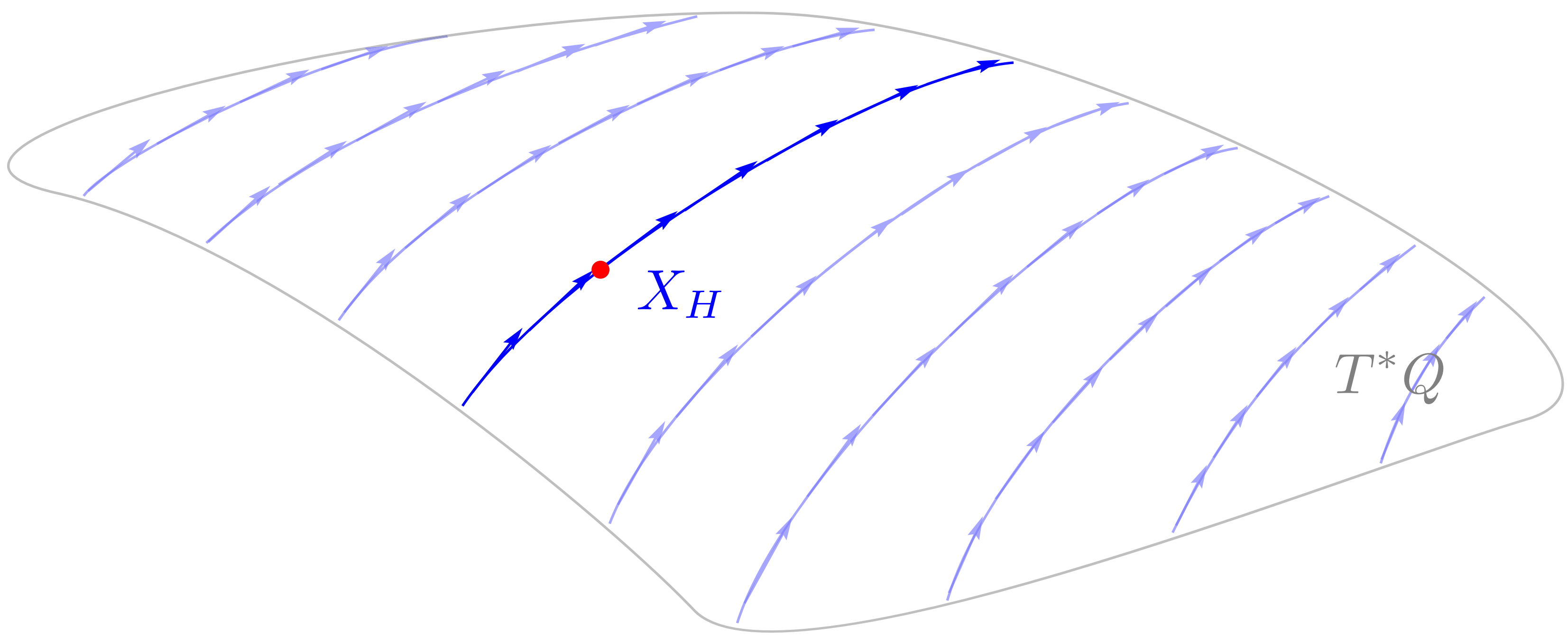
↓ unpolarized radiation

light transport equation

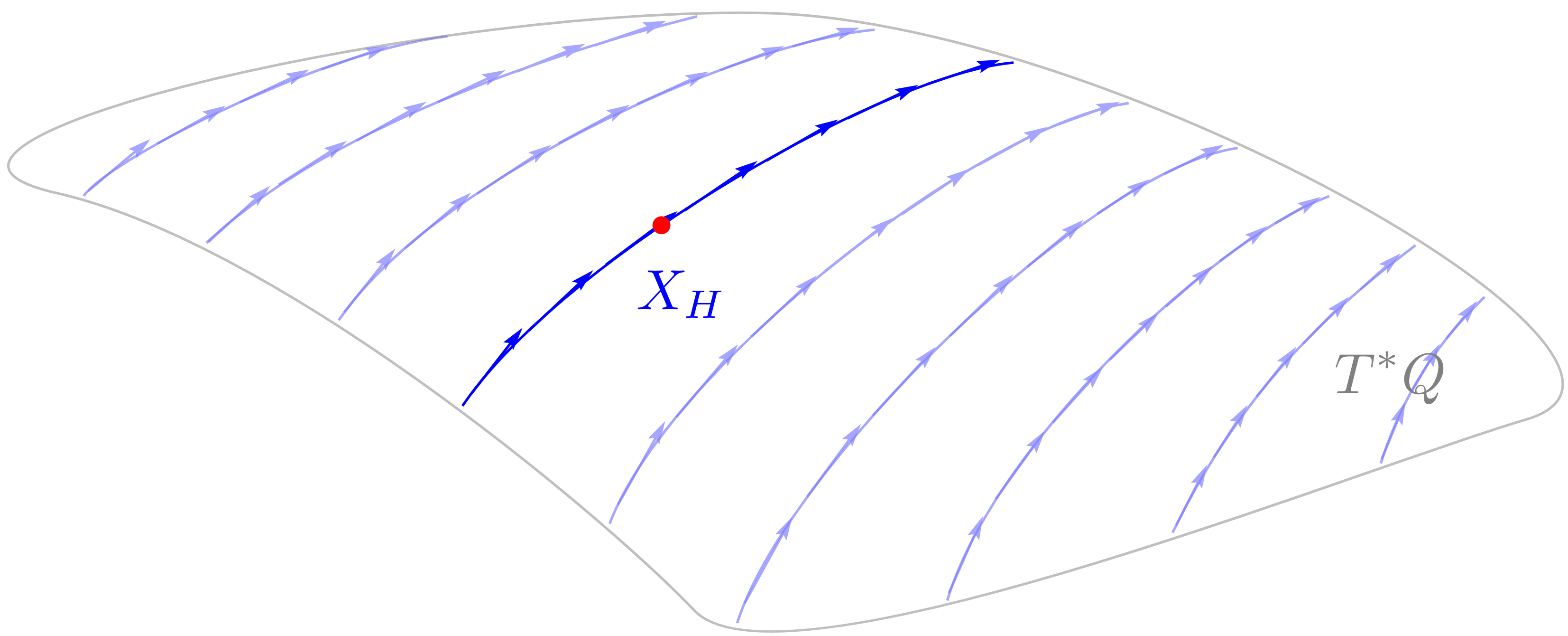
$$\dot{\ell} = -\{\ell, H\}$$

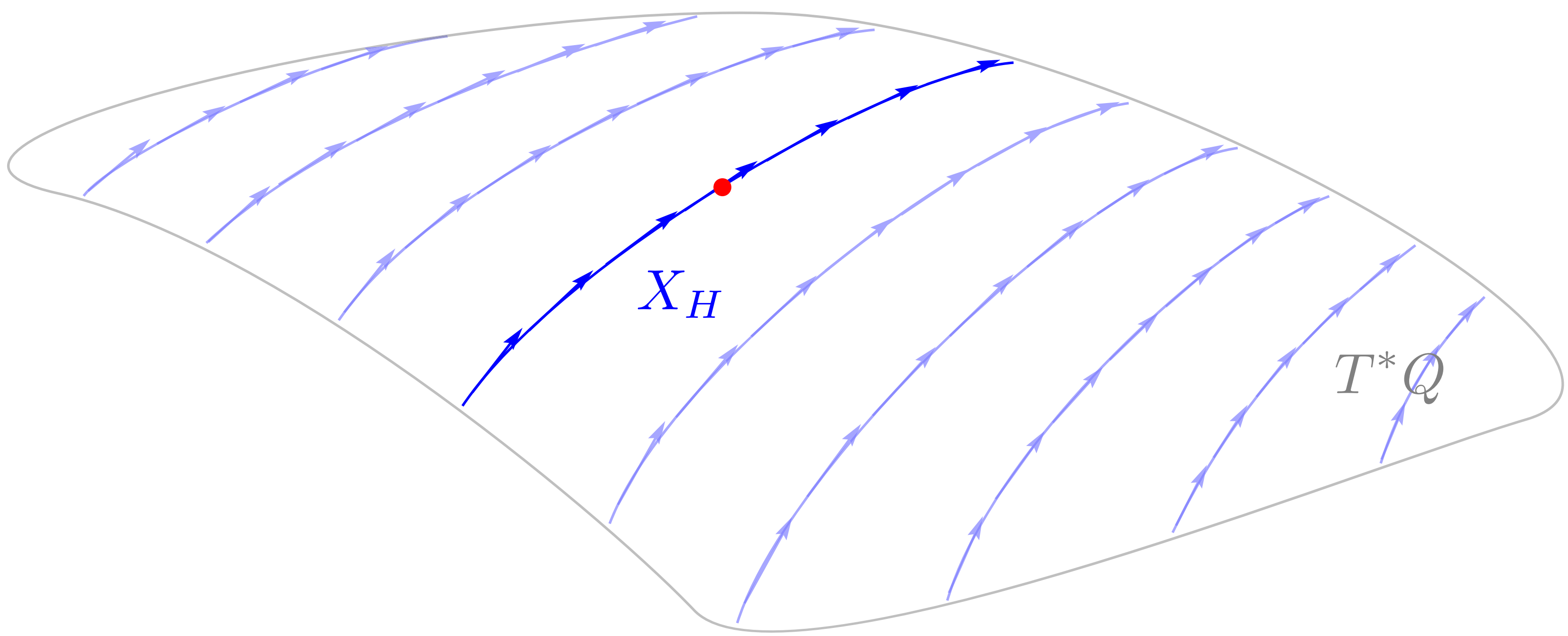


X_H
T^*Q

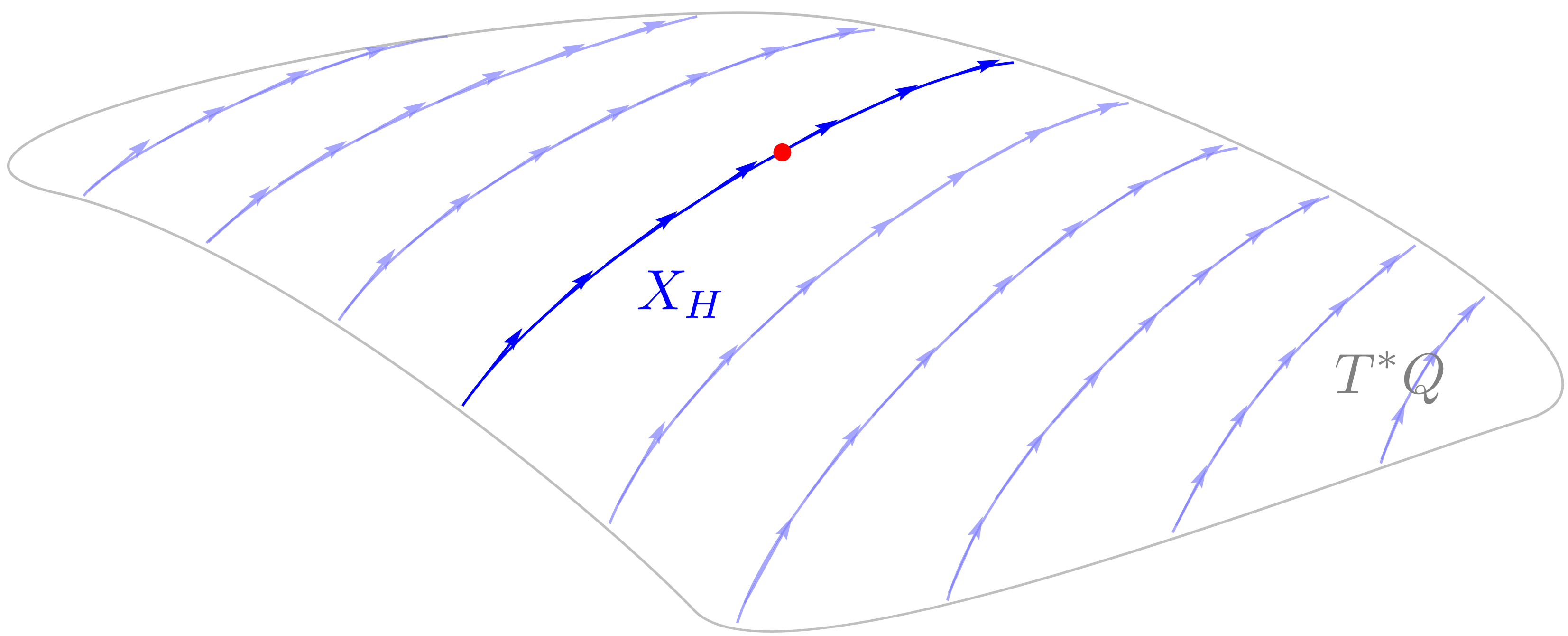


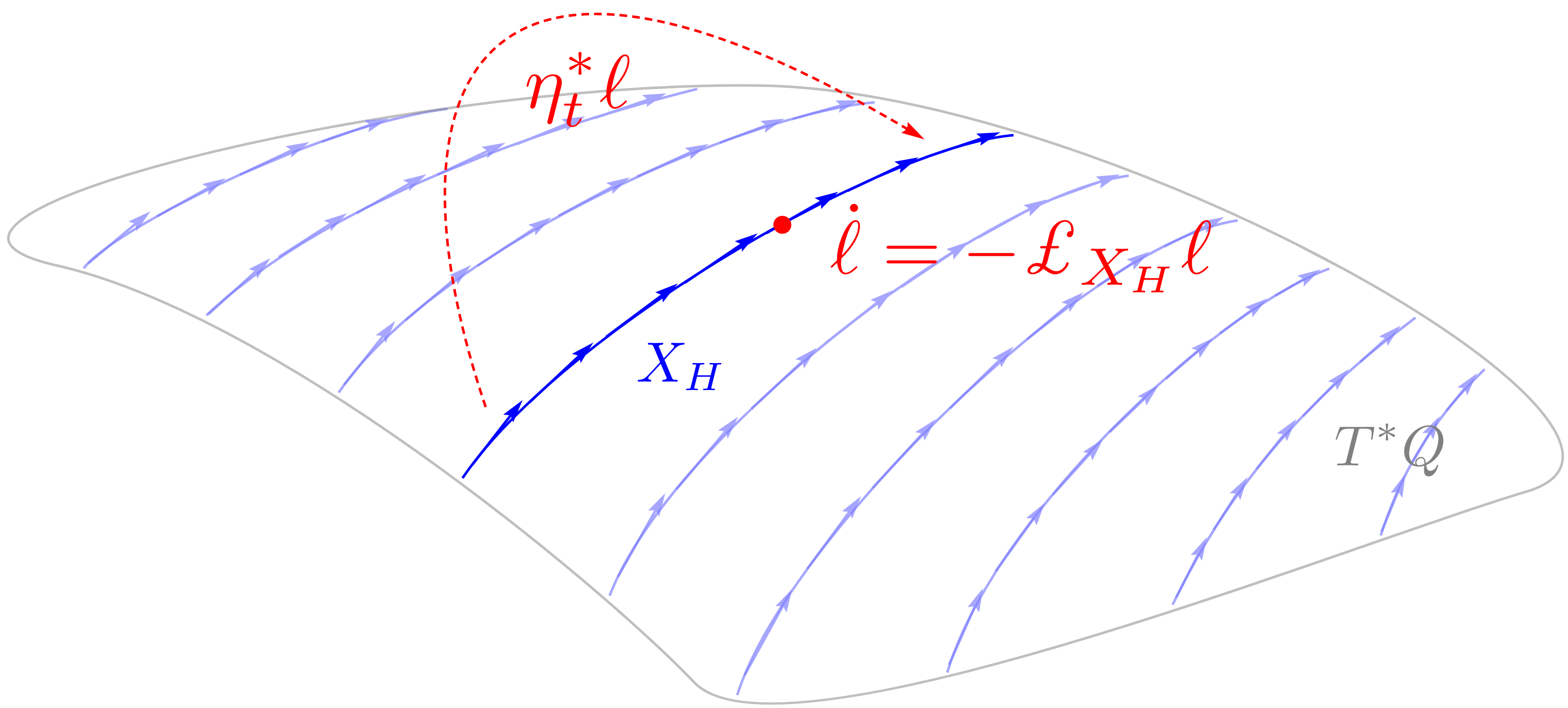
X_H
T^*Q


X_H
T^*Q



X_H
T^*Q


X_H
T^*Q



$\eta_t^* \ell$
$\dot{\ell} = -\pounds_{X_H} \ell$
X_H
T^*Q

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↓ microlocal analysis (Wigner transform)

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↓ $\varepsilon \to 0$

$$\dot{W}_a^0 + \left\{\tau_a, W_a^0\right\} = \left[W_a^0, F_a\right]$$

↓ unpolarized radiation

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

configuration space $Q \subset \mathbb{R}^3$

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↓ unpolarized radiation

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

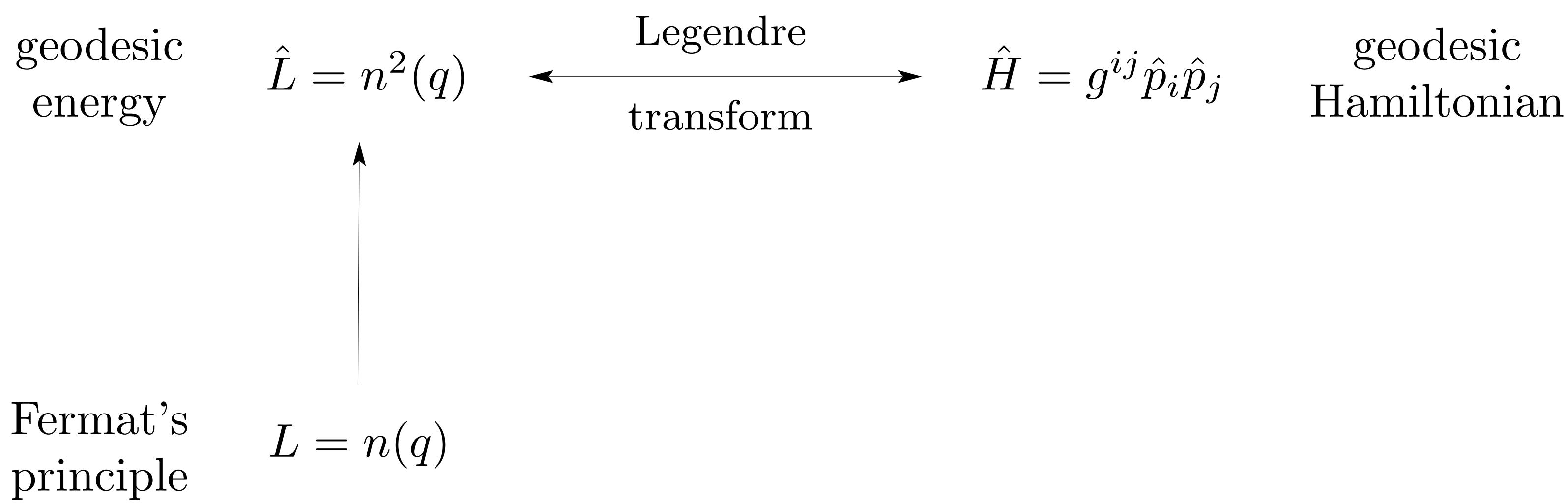
$$H(q,p) = \frac{c}{n(q)}\,\|p\|$$

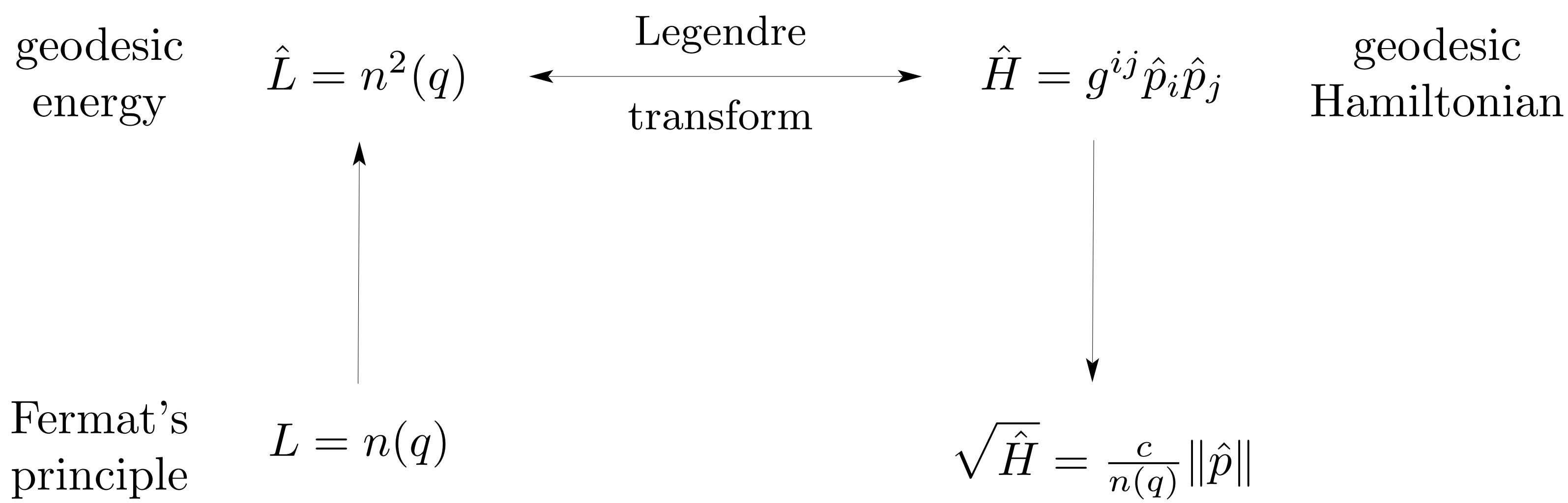
Fermat's principle $\quad L = n(q)$

geodesic energy $\hat{L} = n^2(q)$

$\uparrow$

Fermat's principle $L = n(q)$


geodesic energy
$\hat{L} = n^2(q)$
Legendre transform
$\hat{H} = g^{ij}\hat{p}_i\hat{p}_j$
geodesic Hamiltonian
Fermat's principle
$L = n(q)$


geodesic energy
$\hat{L} = n^2(q)$
Legendre transform
$\hat{H} = g^{ij}\hat{p}_i\hat{p}_j$
geodesic Hamiltonian
Fermat's principle
$L = n(q)$
$\sqrt{\hat{H}} = \frac{c}{n(q)}\|\hat{p}\|$

geodesic energy $\hat{L} = n^2(q)$ ⟷ (Legendre transform) $\hat{H} = g^{ij}\hat{p}_i\hat{p}_j$ geodesic Hamiltonian

↑ (from Fermat's principle to geodesic energy)

Fermat's principle $L = n(q)$

↓ (from geodesic Hamiltonian)

$$\sqrt{\hat{H}} = \frac{c}{n(q)}\|\hat{p}\|$$

↓

$$H = \omega\check{H} = \frac{c}{n(q)}\|p\|$$

light transport

geodesic energy	$\hat{L} = n^2(q)$	⟵ Legendre transform ⟶	$\hat{H} = g^{ij}\hat{p}_i\hat{p}_j$	geodesic Hamiltonian
	↑		↓	
Fermat's principle	$L = n(q)$		$\sqrt{\hat{H}} = \frac{c}{n(q)}\lVert\hat{p}\rVert$	
	↑		↓	
	$L(q,\dot{q}) = \omega\sqrt{g^n_{ij}\dot{q}^i\,\dot{q}^j}$		$H = \omega\check{H} = \frac{c}{n(q)}\lVert p\rVert$	light transport

configuration space $Q \subset \mathbb{R}^3$

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{\epsilon}\nabla\times\\ \frac{1}{\mu}\nabla\times & 0\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$$

↓ microlocal analysis (Wigner transform)

phase space T^*Q

$$-\dot{W}^{\varepsilon} = \frac{1}{\varepsilon}\left[p^{\varepsilon}, W^{\varepsilon}\right] + \frac{1}{i}\left\{p^{\varepsilon}, W^{\varepsilon}\right\} + O(\varepsilon)$$

↓ $\varepsilon \to 0$

$$\dot{W}_a^0 + \left\{\tau_a, W_a^0\right\} = \left[W_a^0, F_a\right]$$

↓ unpolarized radiation

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

← Legendre transform

Fermat's principle

$$\hat{L} = n^2(q)$$

electromagnetic theory

configuration space $Q \subset \mathbb{R}^3$

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↓ $\varepsilon \to 0$

$$\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]$$

conservation of frequency

↓ unpolarized radiation

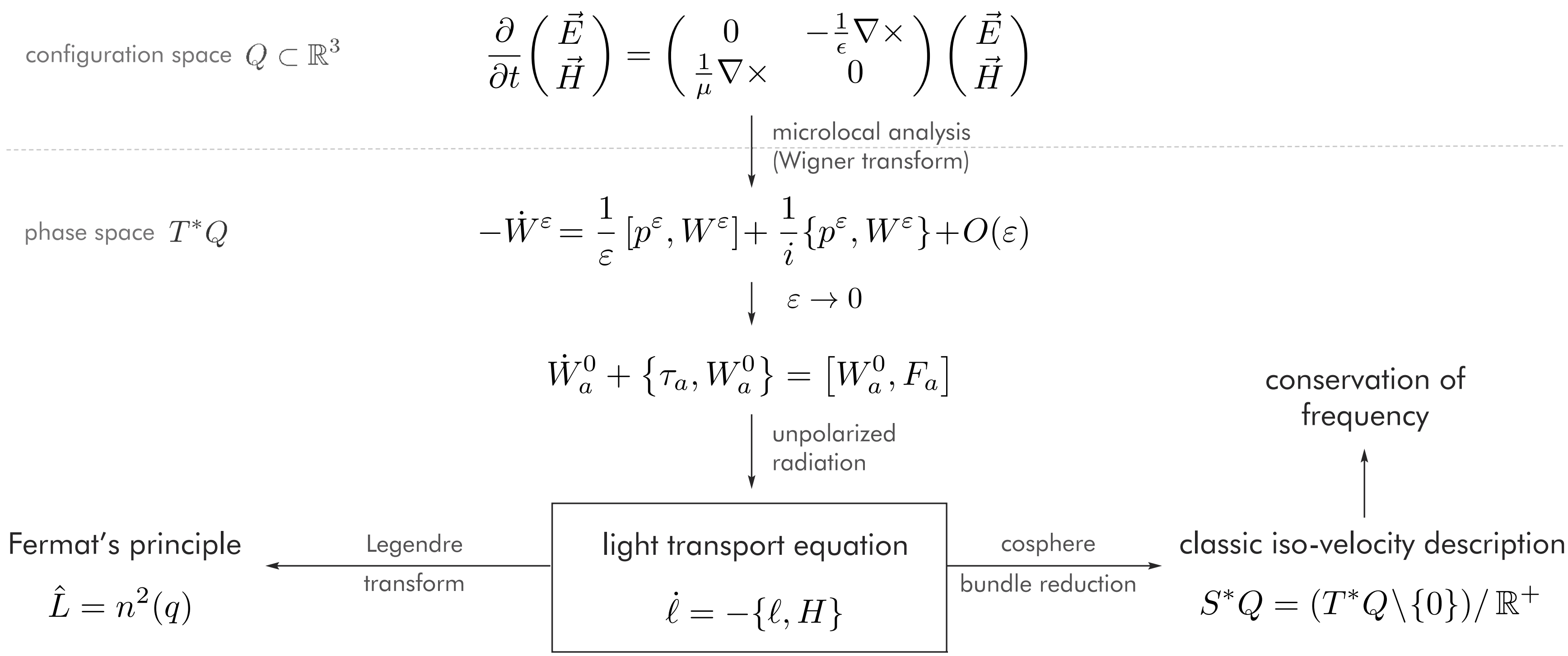
light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

← Legendre transform

Fermat's principle

$$\hat{L} = n^2(q)$$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microlocal analysis
(Wigner transform)
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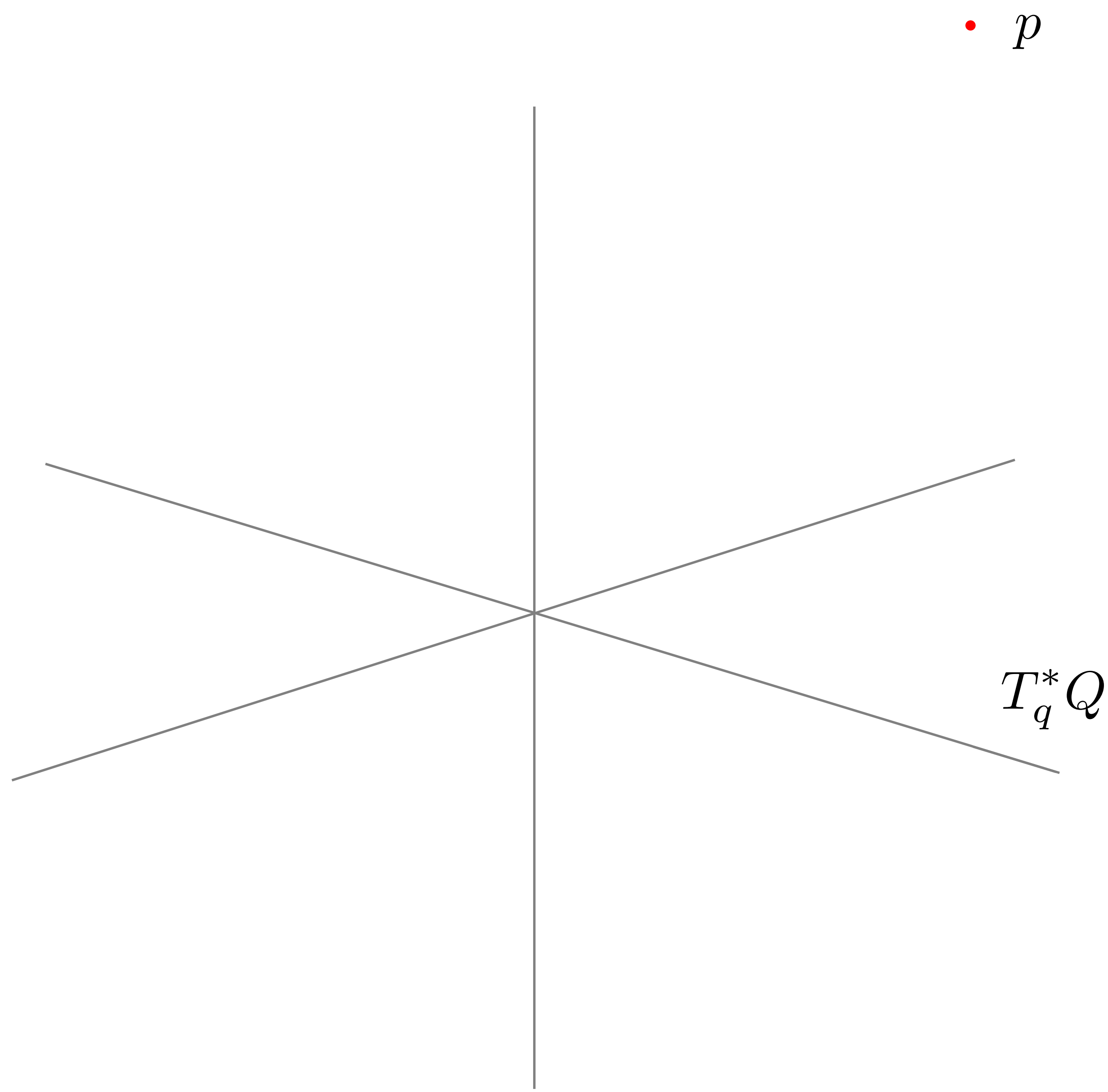
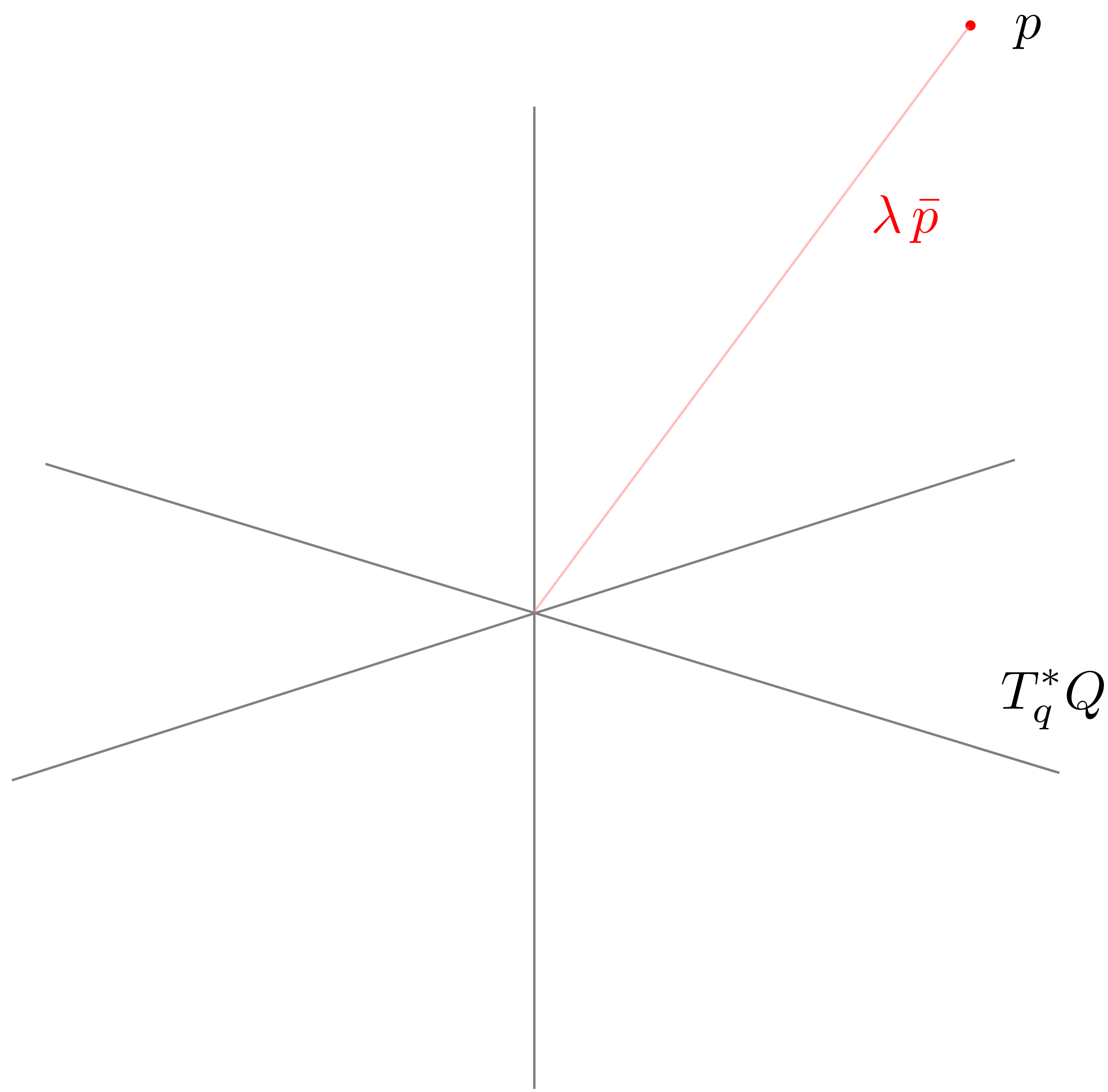
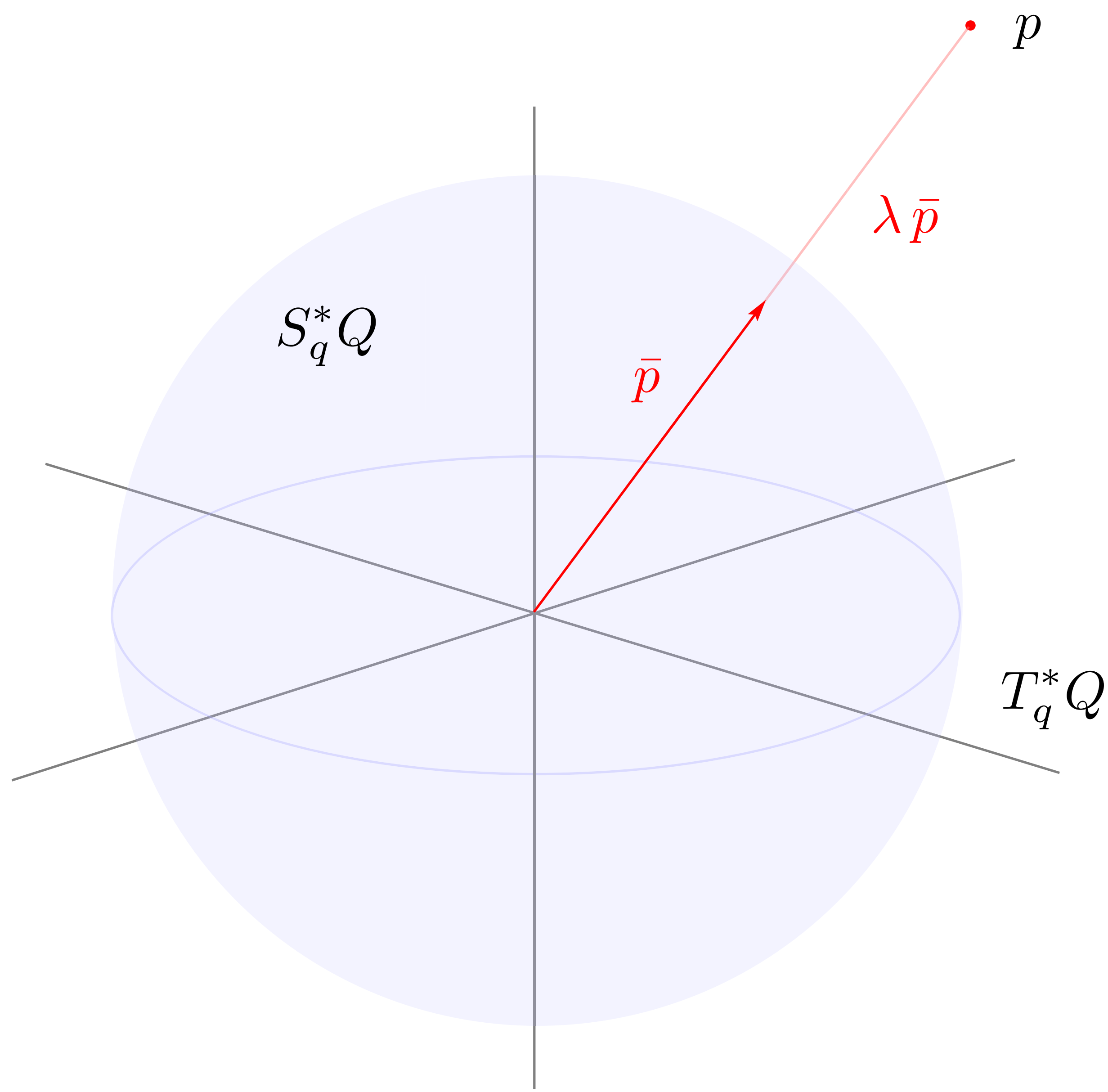
$\varepsilon \to 0$
$\dot{W}_a^0 + \{\tau_a, W_a^0\} = [W_a^0, F_a]$
unpolarized
radiation
conservation of
frequency
Fermat's principle
$\hat{L} = n^2(q)$
Legendre
transform
light transport equation
$\dot{\ell} = -\{\ell, H\}$
cosphere
bundle reduction
classic iso-velocity description
$S^*Q = (T^*Q \backslash \{0\}) / \mathbb{R}^+$

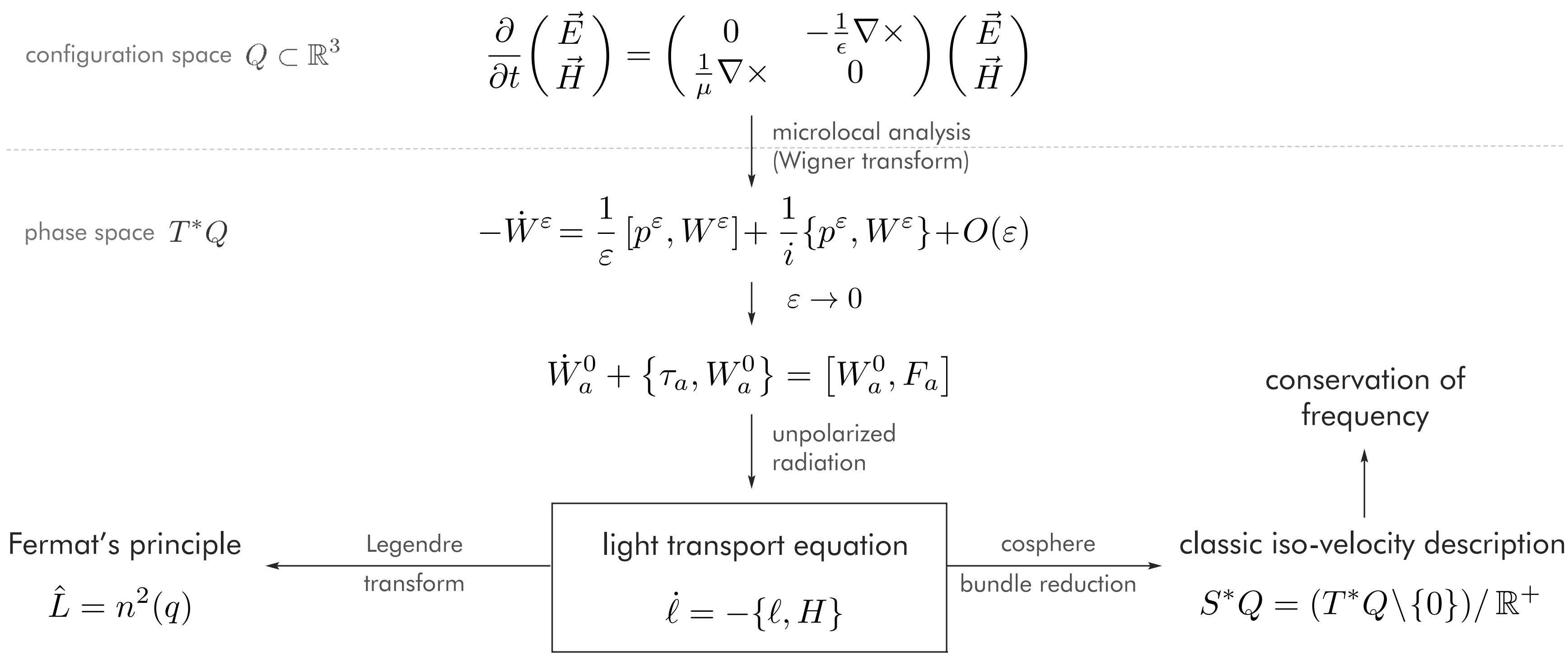


p
T_q^*Q


p
$\lambda\,\bar{p}$
T_q^*Q



p
$\lambda \bar{p}$
S_q^*Q
$\bar{p}$
T_q^*Q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microlocal analysis
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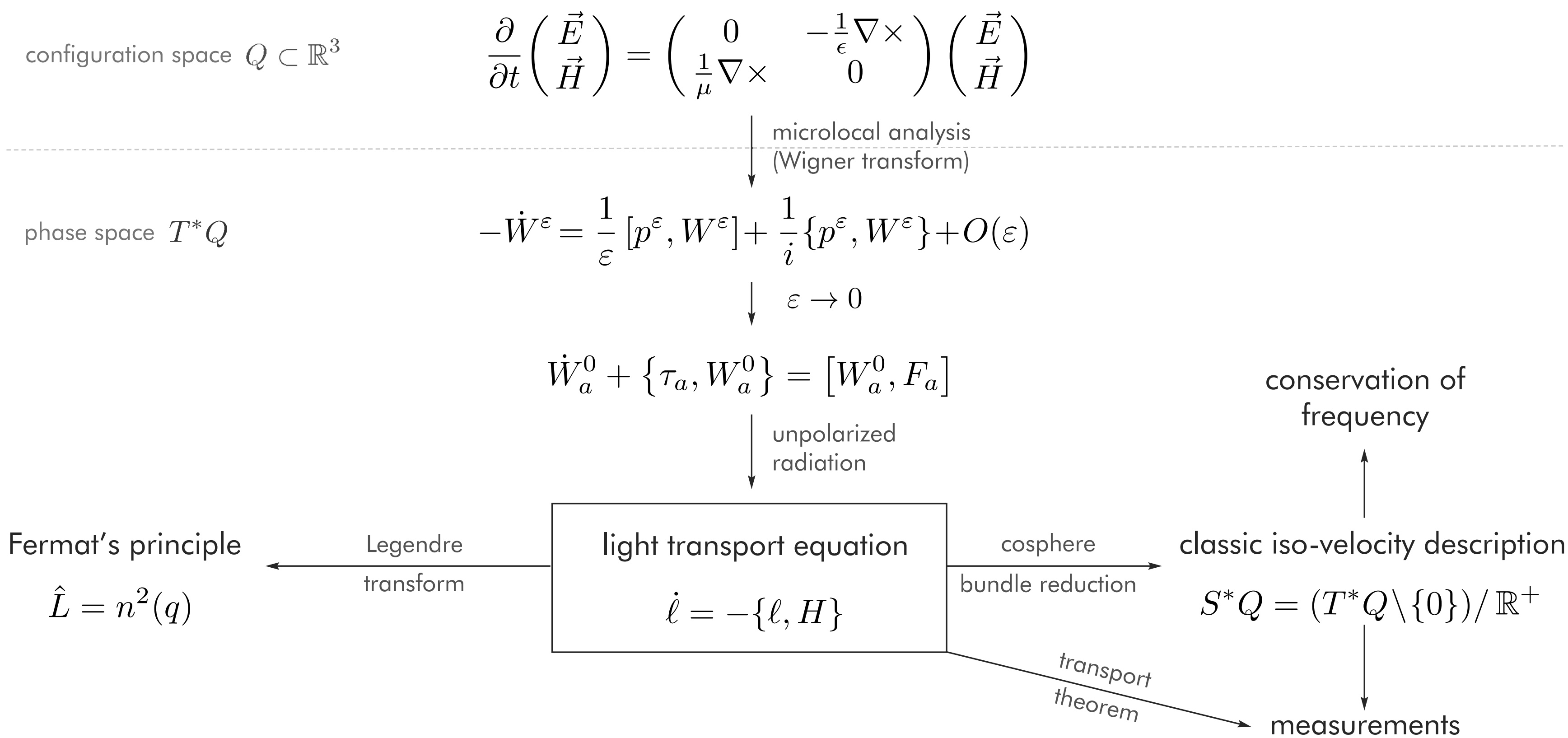
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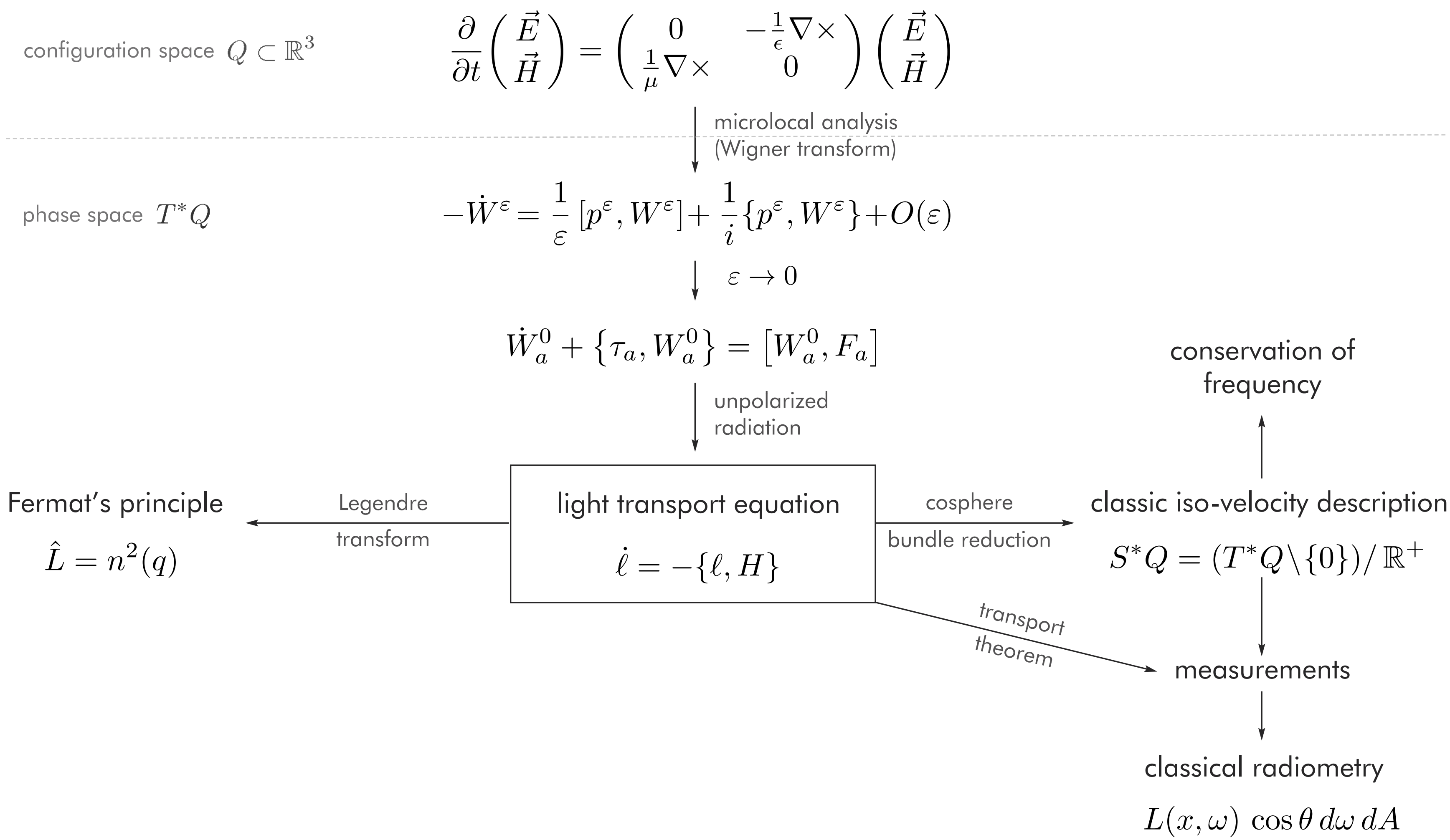
$S^*Q = (T^*Q \backslash \{0\}) / \mathbb{R}^+$
conservation of
frequency
transport
theorem
measurements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$\hat{L} = n^2(q)$
Legendre
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light transport equation
$\dot{\ell} = -\{\ell, H\}$
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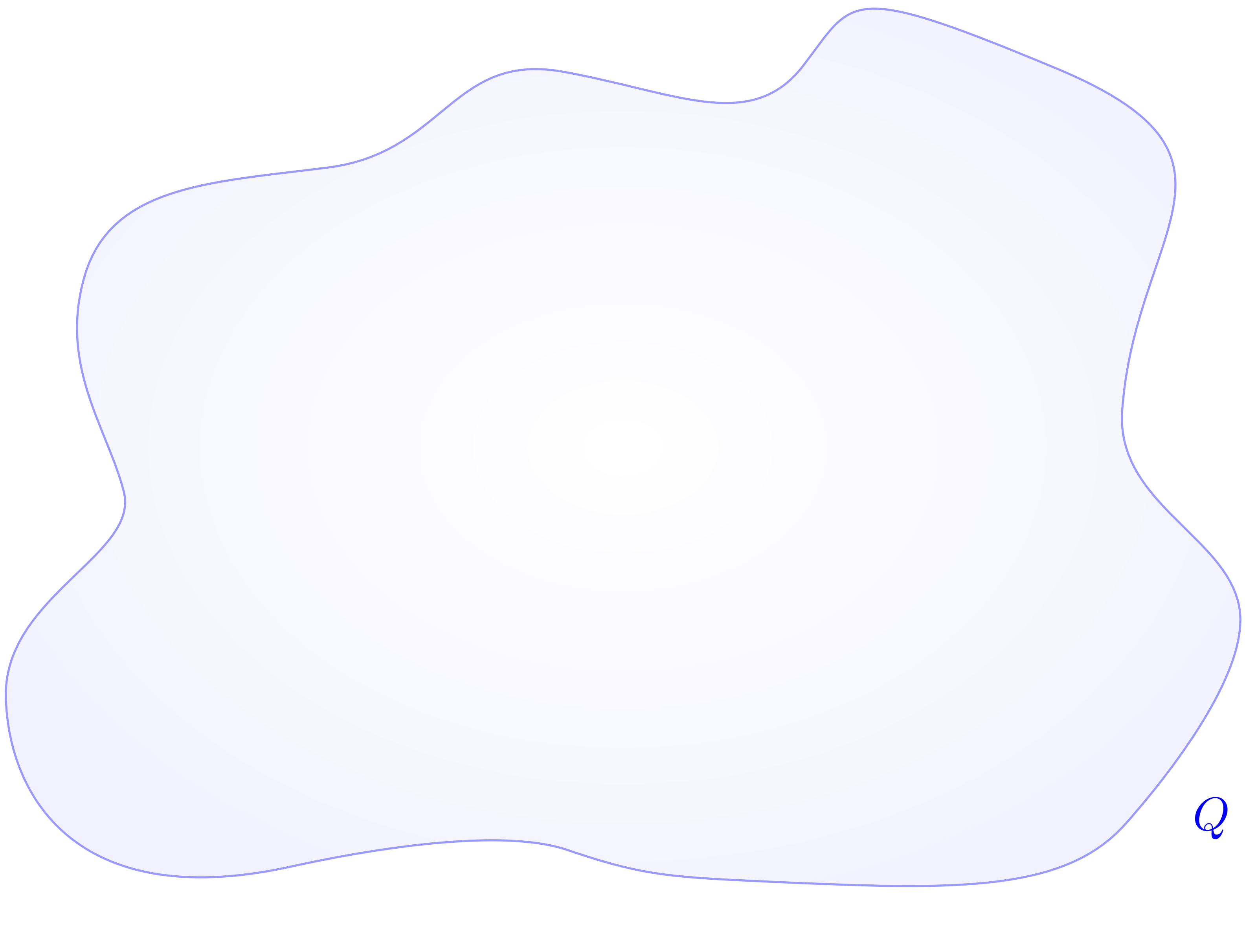
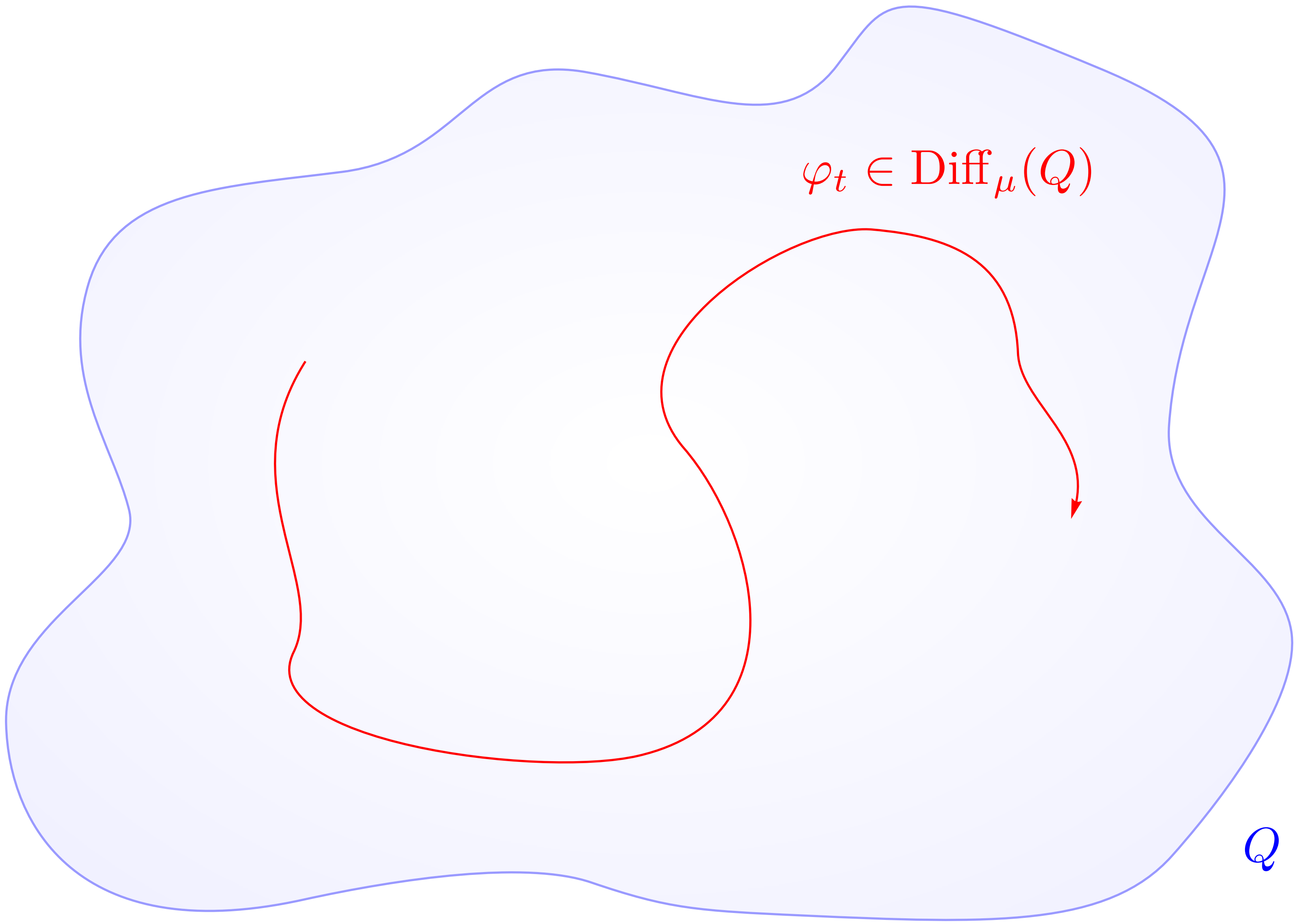
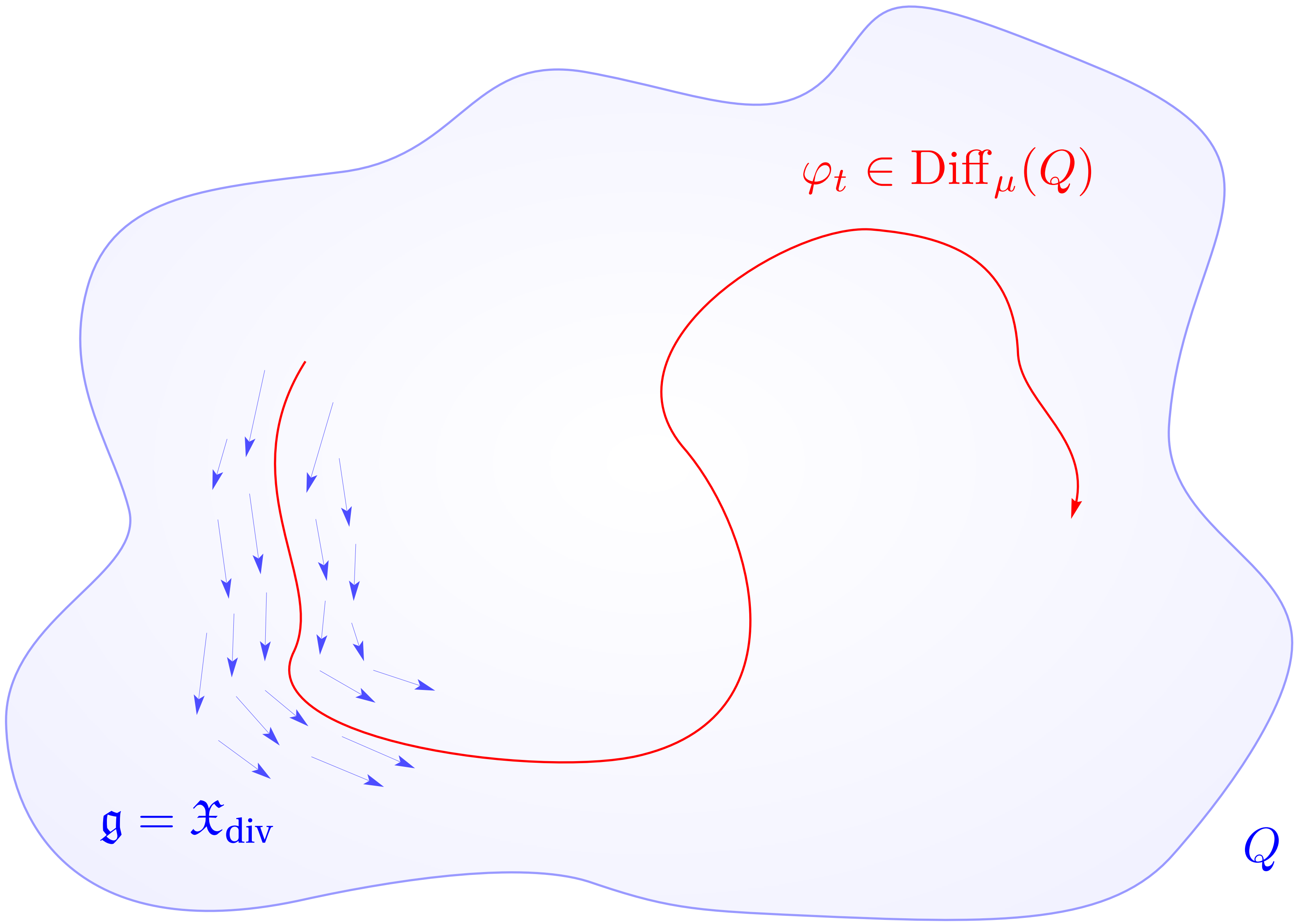
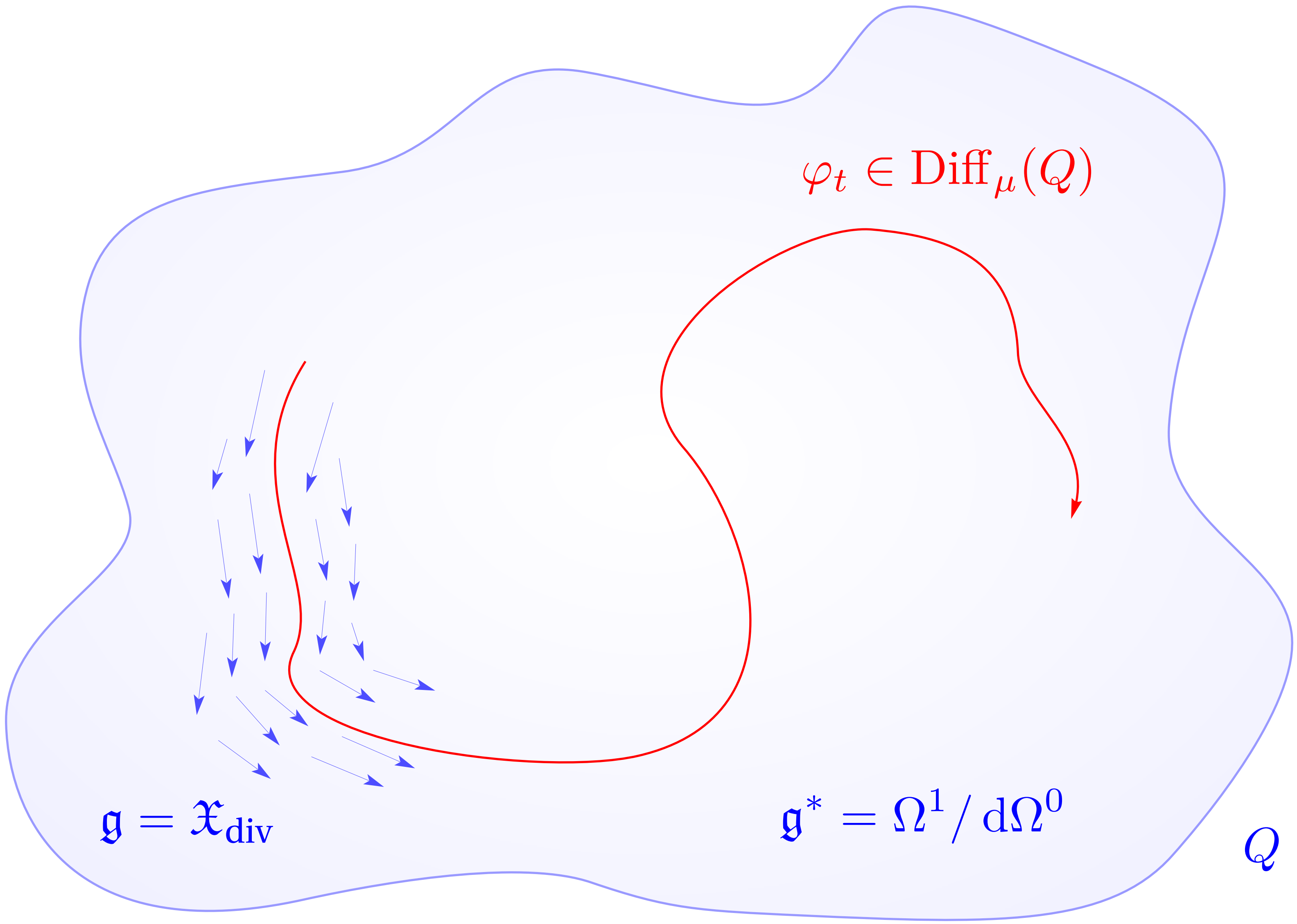
bundle reduction
classic iso-velocity description
$S^*Q = (T^*Q \backslash \{0\}) / \mathbb{R}^+$
transport
theorem
measurements
classical radiometry
$L(x, \omega)\ \cos\theta\, d\omega\, dA$

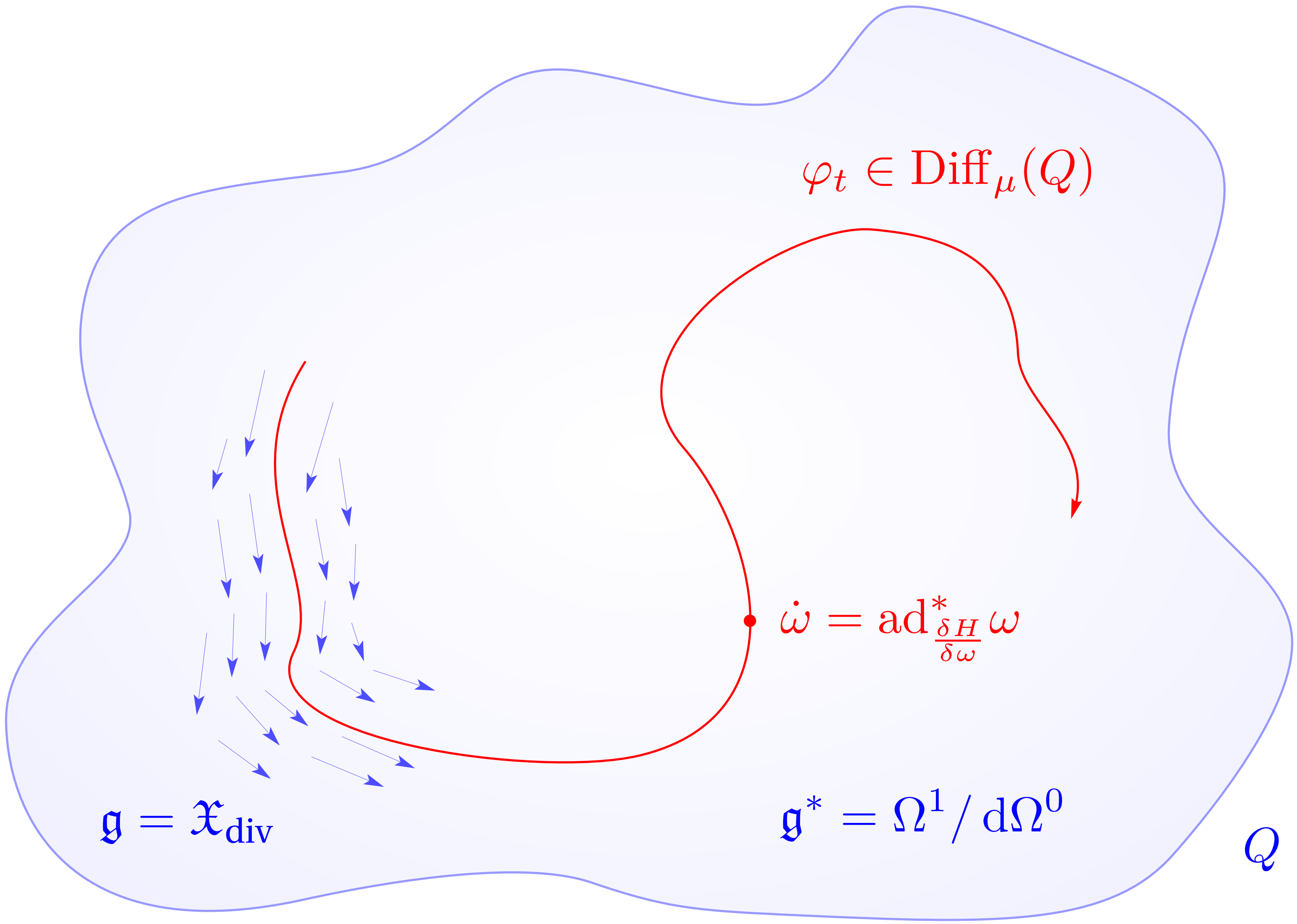

Q


$\varphi_t \in \mathrm{Diff}_\mu(Q)$
Q


$\varphi_t \in \mathrm{Diff}_\mu(Q)$
$\mathfrak{g} = \mathfrak{X}_{\mathrm{div}}$
Q



$\varphi_t \in \mathrm{Diff}_\mu(Q)$
$\mathfrak{g} = \mathfrak{X}_{\mathrm{div}}$
$\mathfrak{g}^* = \Omega^1 / \mathrm{d}\Omega^0$
Q



$\varphi_t \in \mathrm{Diff}_\mu(Q)$
$\dot{\omega} = \mathrm{ad}^*_{\frac{\delta H}{\delta \omega}} \omega$
$\mathfrak{g} = \mathfrak{X}_{\mathrm{div}}$
$\mathfrak{g}^* = \Omega^1 / \mathrm{d}\Omega^0$
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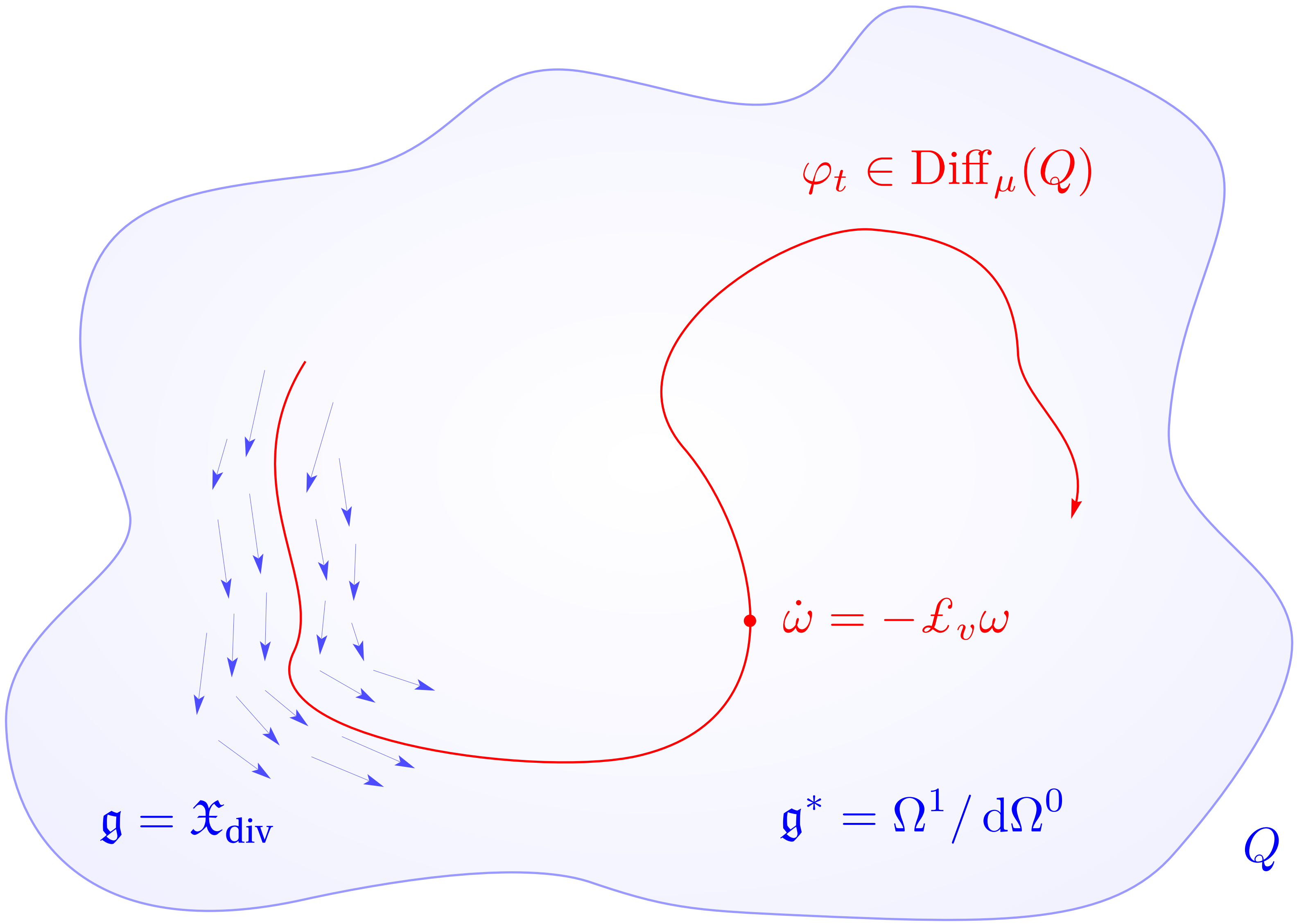
$\varphi_t \in \mathrm{Diff}_\mu(Q)$
$\dot\omega = -\pounds_v \omega$
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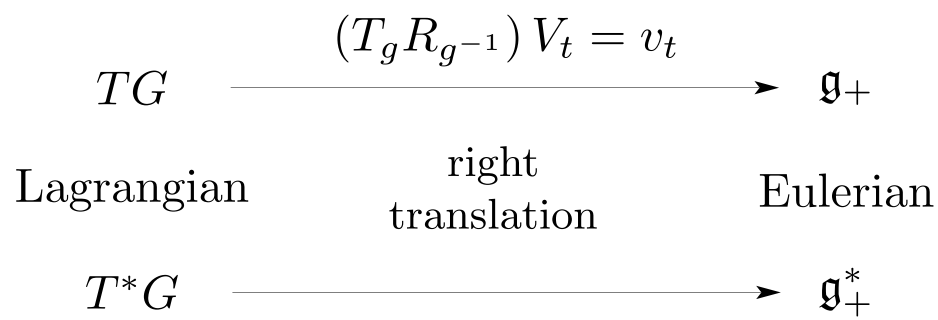
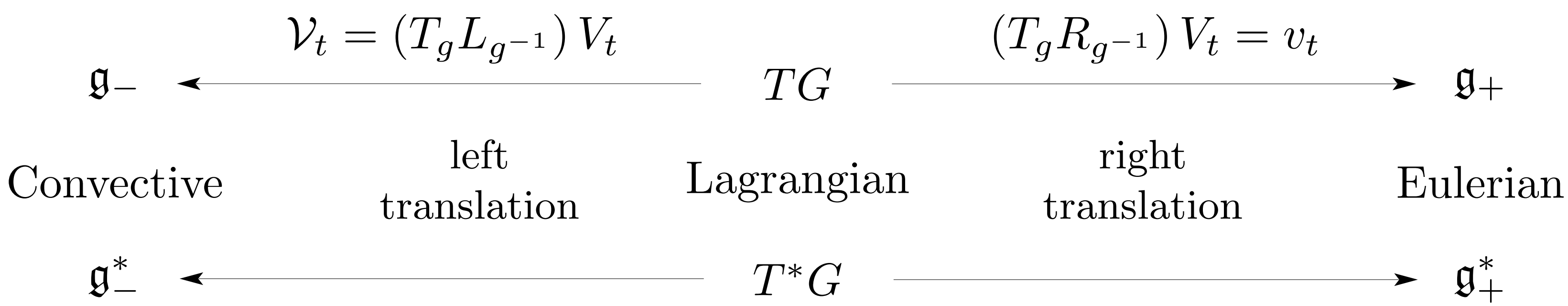
$\mathfrak{g}^* = \Omega^1 / \mathrm{d}\Omega^0$
Q

TG

Lagrangian

T^*G


$(T_g R_{g^{-1}}) V_t = v_t$
TG
$\mathfrak{g}_+$
Lagrangian
right translation
Eulerian
T^*G
$\mathfrak{g}^*_+$



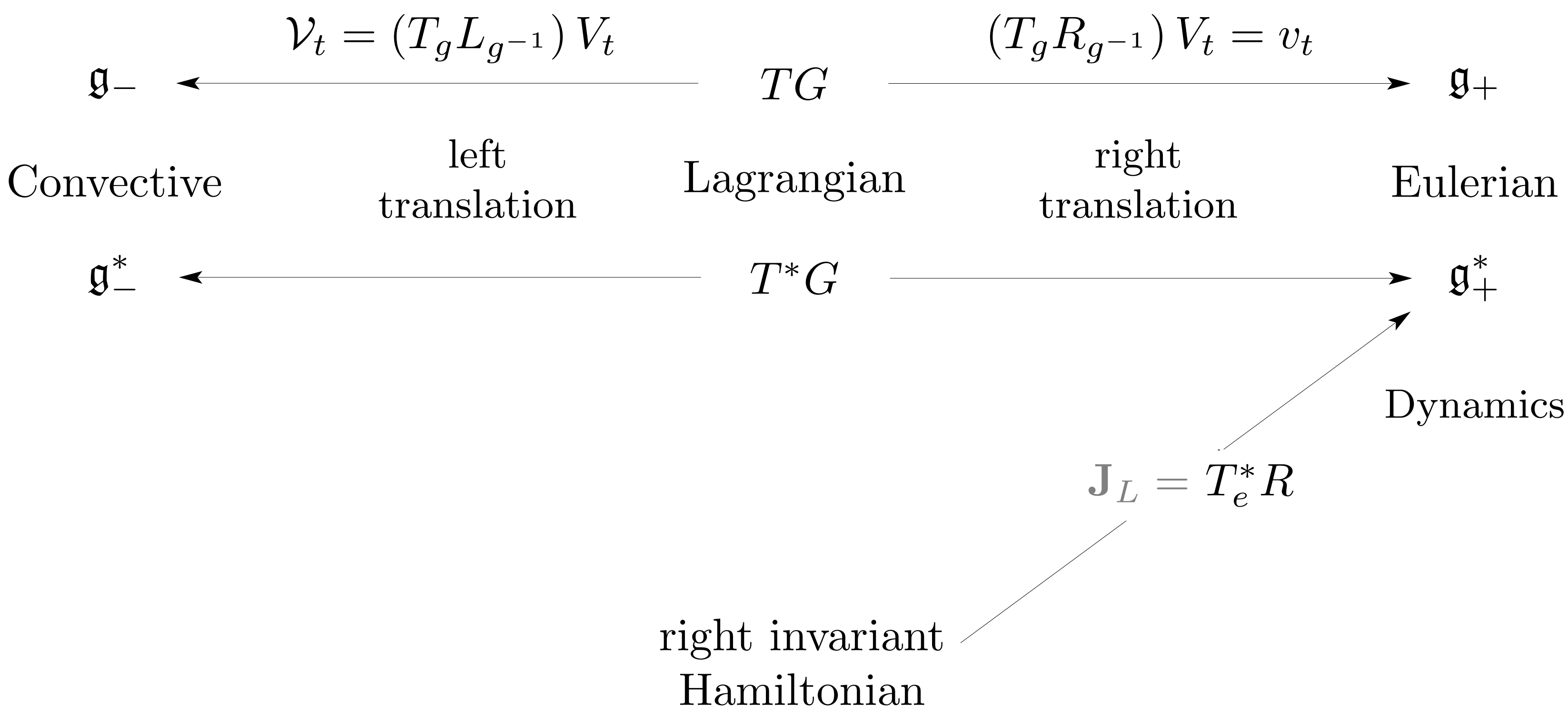
$\mathcal{V}_t = (T_g L_{g^{-1}})\, V_t$
$(T_g R_{g^{-1}})\, V_t = v_t$
$\mathfrak{g}_-$
TG
$\mathfrak{g}_+$
Convective
left translation
Lagrangian
right translation
Eulerian
$\mathfrak{g}^*_-$
T^*G
$\mathfrak{g}^*_+$

$$\mathfrak{g}_- \xleftarrow{\;\mathcal{V}_t = (T_g L_{g^{-1}})\, V_t\;} TG \xrightarrow{\;(T_g R_{g^{-1}})\, V_t = v_t\;} \mathfrak{g}_+$$

Convective — left translation — Lagrangian — right translation — Eulerian

$$\mathfrak{g}_-^* \longleftarrow T^*G \longrightarrow \mathfrak{g}_+^*$$

right invariant
Hamiltonian



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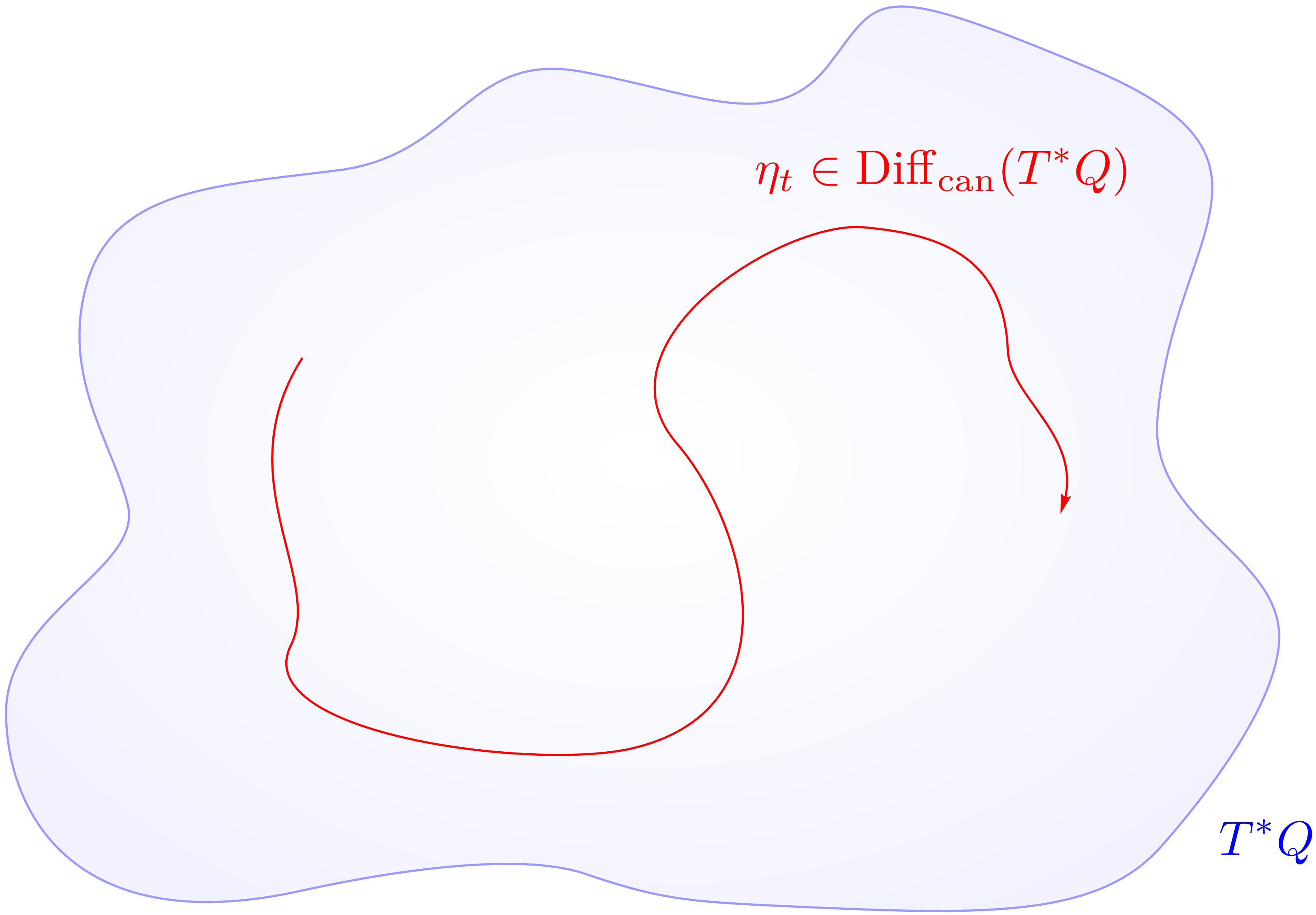
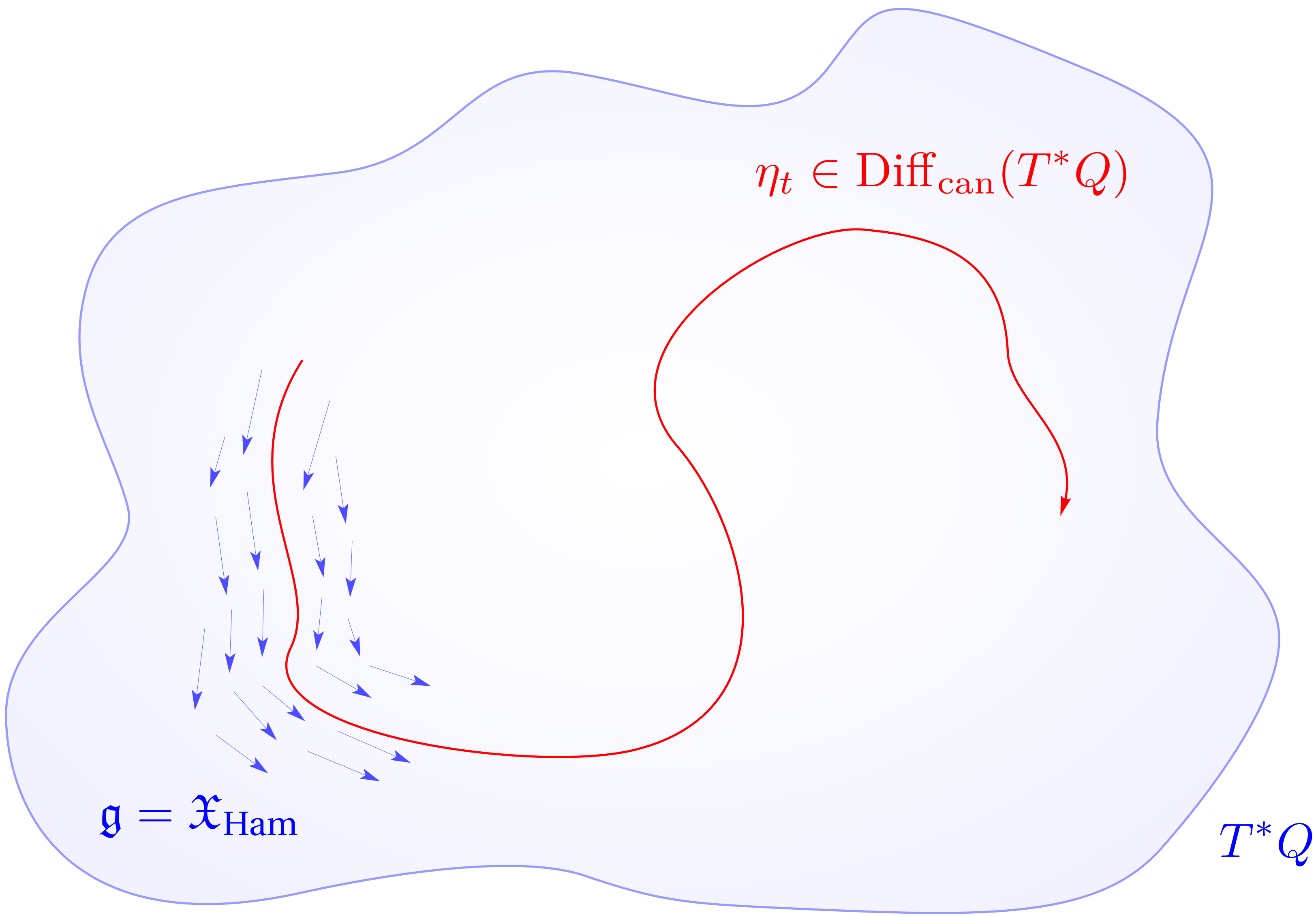
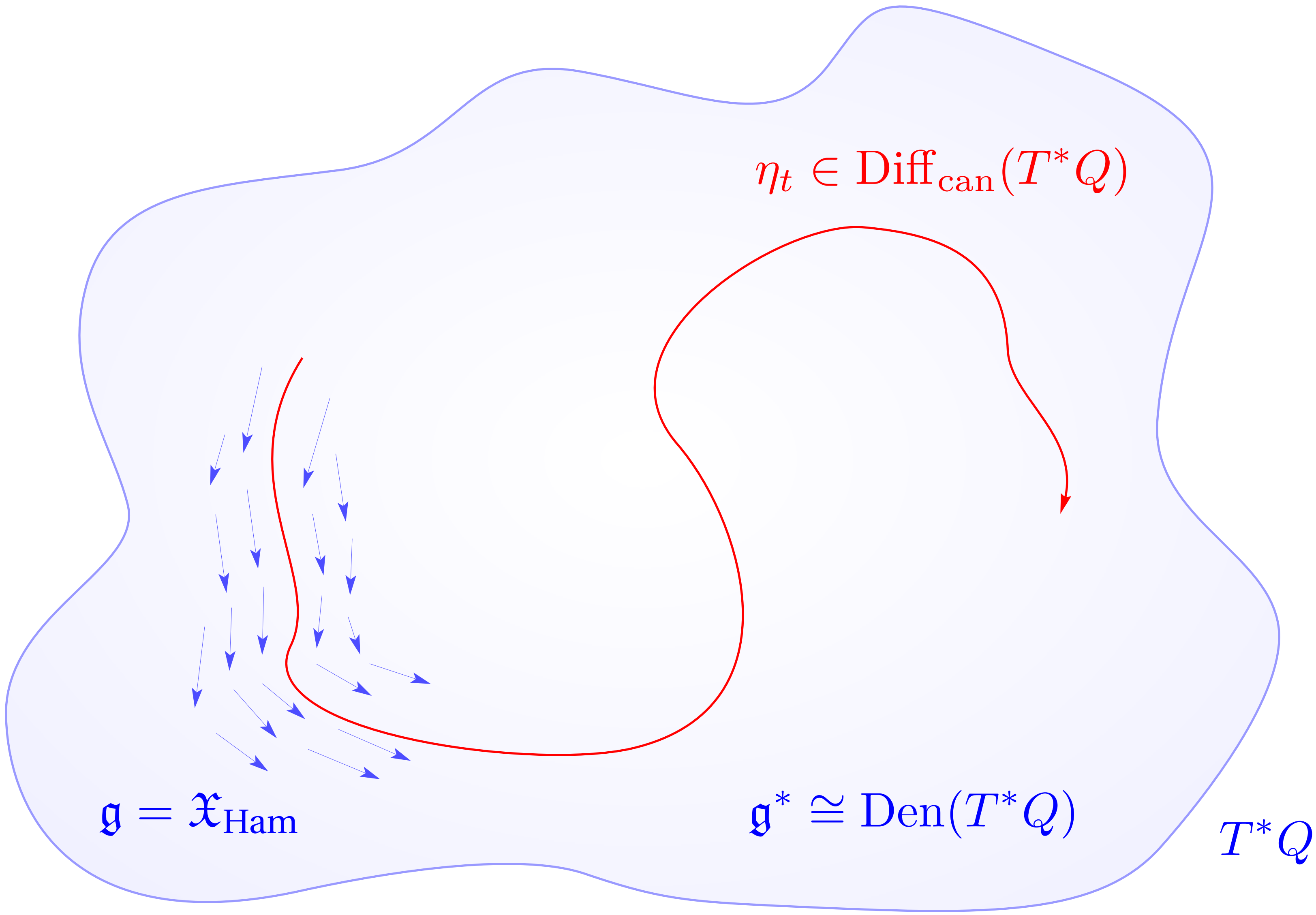
Dynamics
$\mathbf{J}_L = T_e^* R$
right invariant Hamiltonian


$\eta_t \in \mathrm{Diff}_{\mathrm{can}}(T^*Q)$
T^*Q


$\eta_t \in \mathrm{Diff}_{\mathrm{can}}(T^*Q)$
$\mathfrak{g} = \mathfrak{X}_{\mathrm{Ham}}$
T^*Q


$\eta_t \in \mathrm{Diff}_{\mathrm{can}}(T^*Q)$
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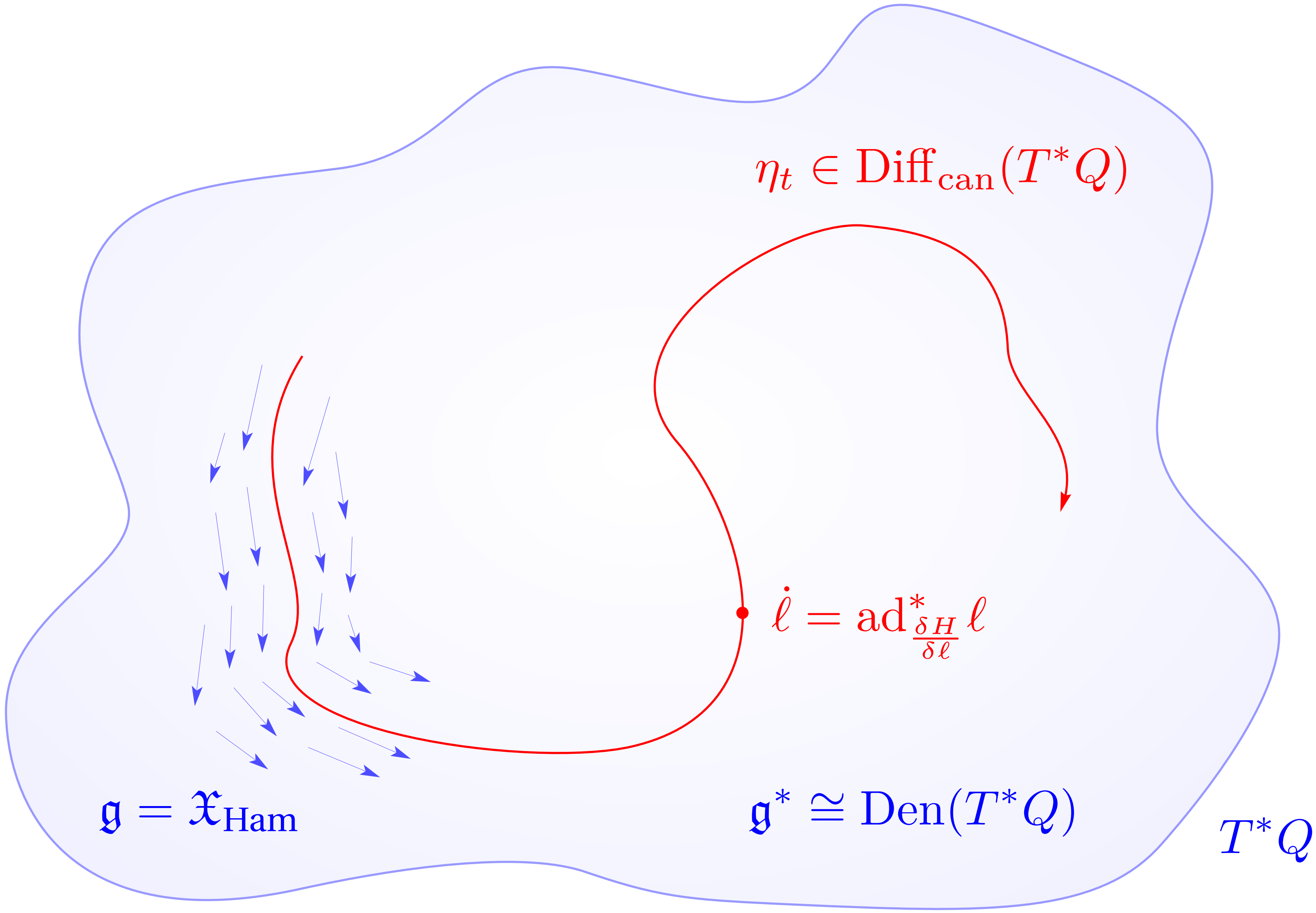
$\mathfrak{g}^* \cong \mathrm{Den}(T^*Q)$
T^*Q


$\eta_t \in \mathrm{Diff}_{\mathrm{can}}(T^*Q)$
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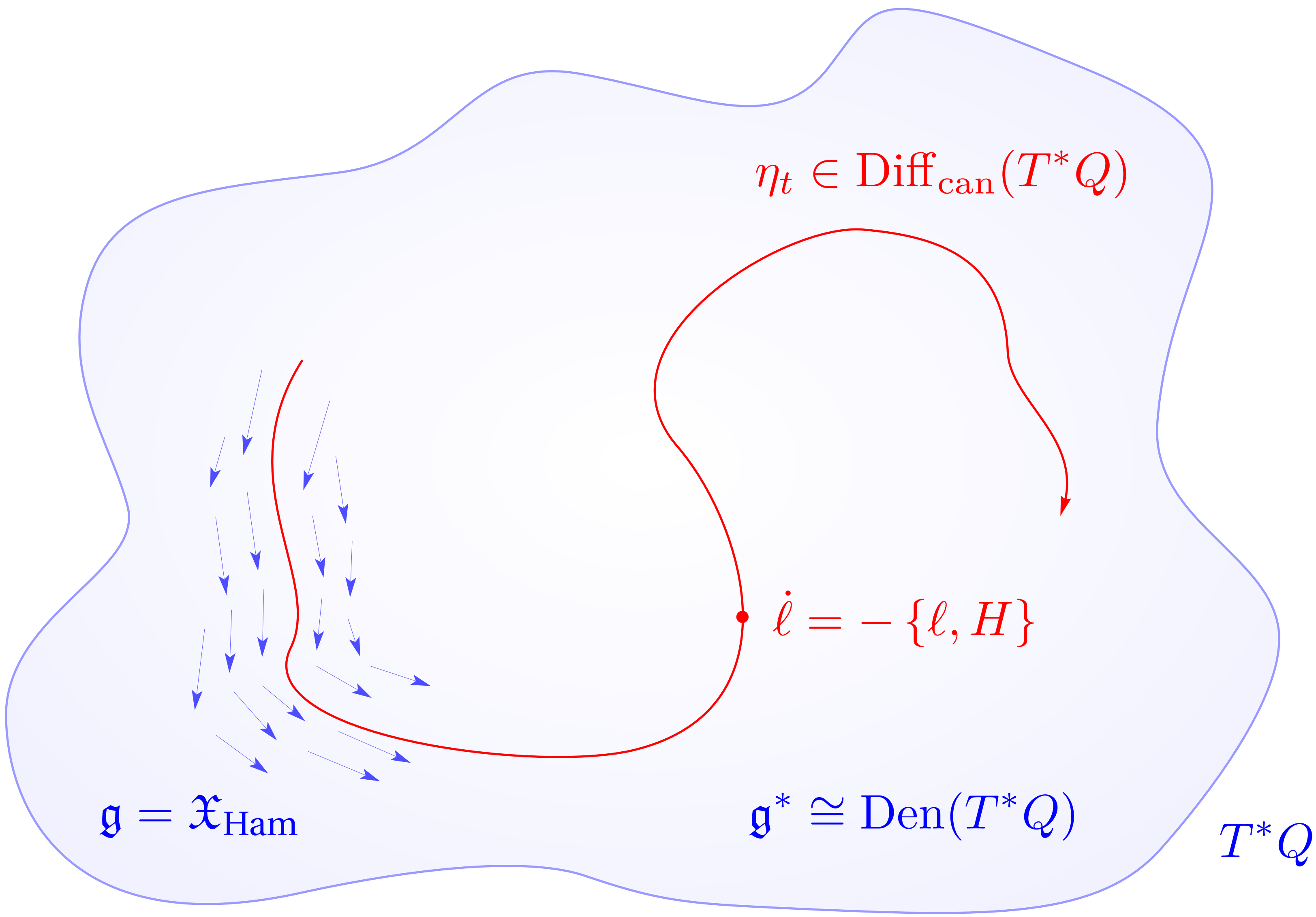
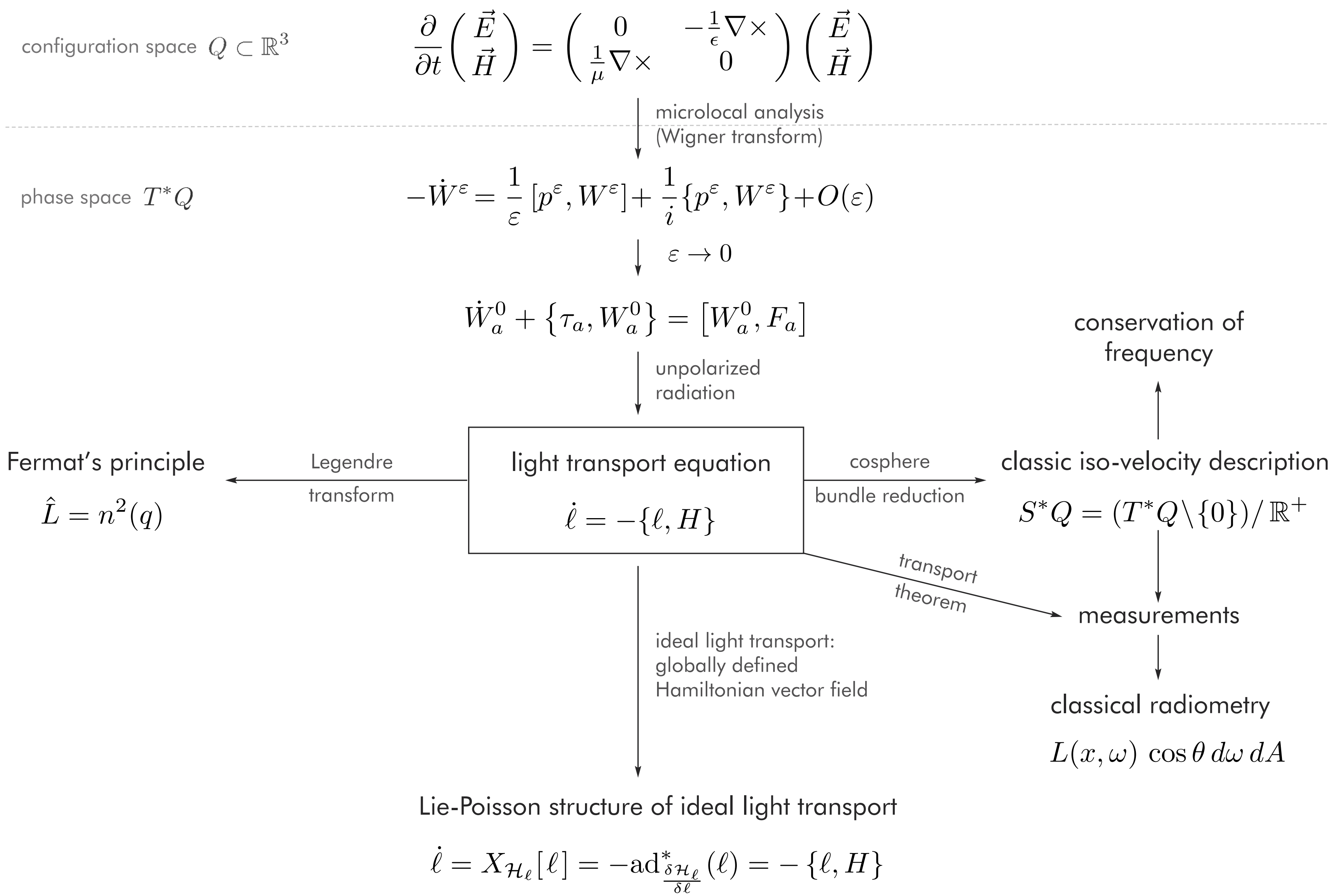
$\mathfrak{g} = \mathfrak{X}_{\mathrm{Ham}}$
$\mathfrak{g}^* \cong \mathrm{Den}(T^*Q)$
T^*Q


$\eta_t \in \mathrm{Diff}_{\mathrm{can}}(T^*Q)$
$\dot{\ell} = -\{\ell, H\}$
$\mathfrak{g} = \mathfrak{X}_{\mathrm{Ham}}$
$\mathfrak{g}^* \cong \mathrm{Den}(T^*Q)$
T^*Q


configuration space $Q \subset \mathbb{R}^3$
$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \begin{pmatrix}0 & -\frac{1}{\epsilon}\nabla\times\\ \frac{1}{\mu}\nabla\times & 0\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}$
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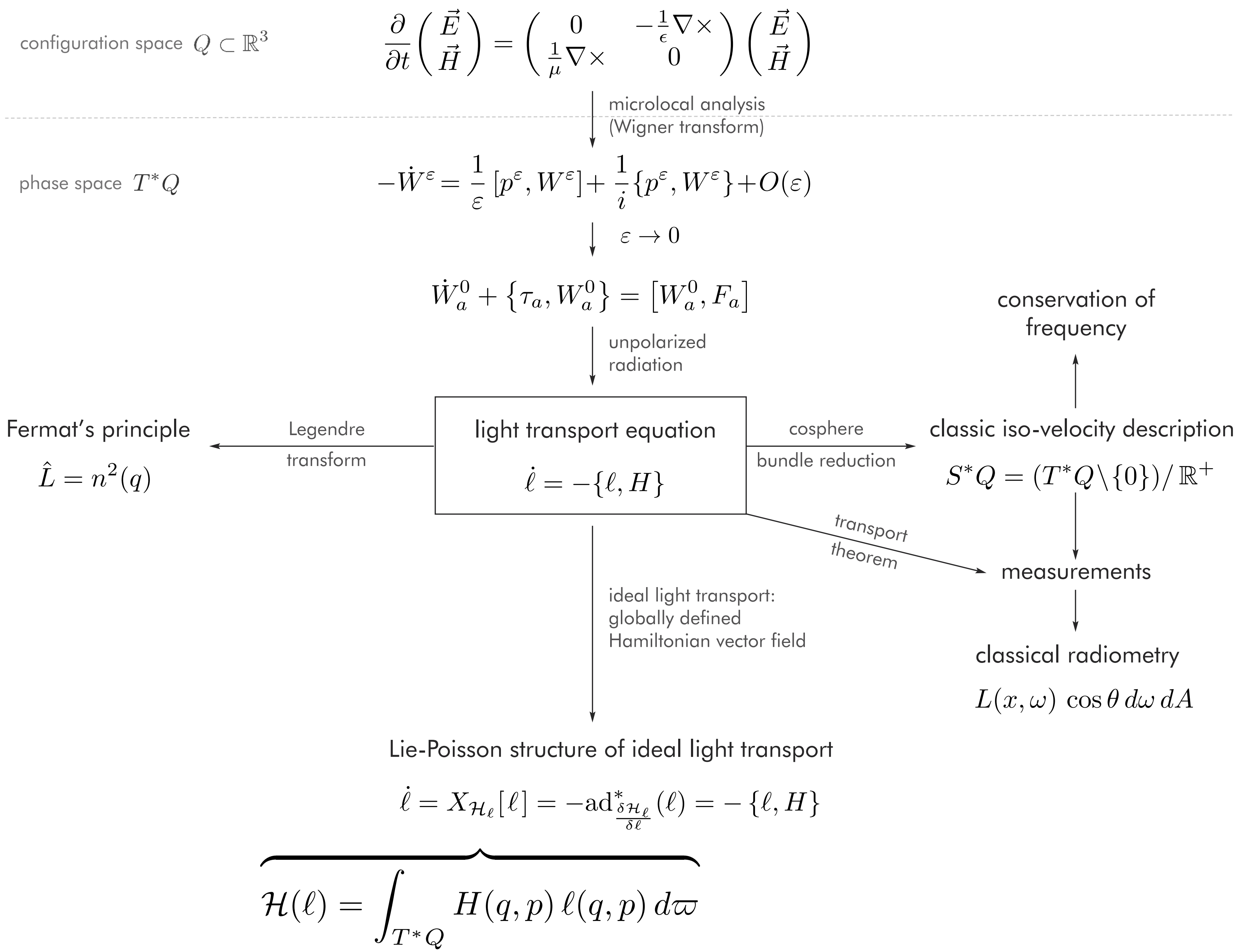
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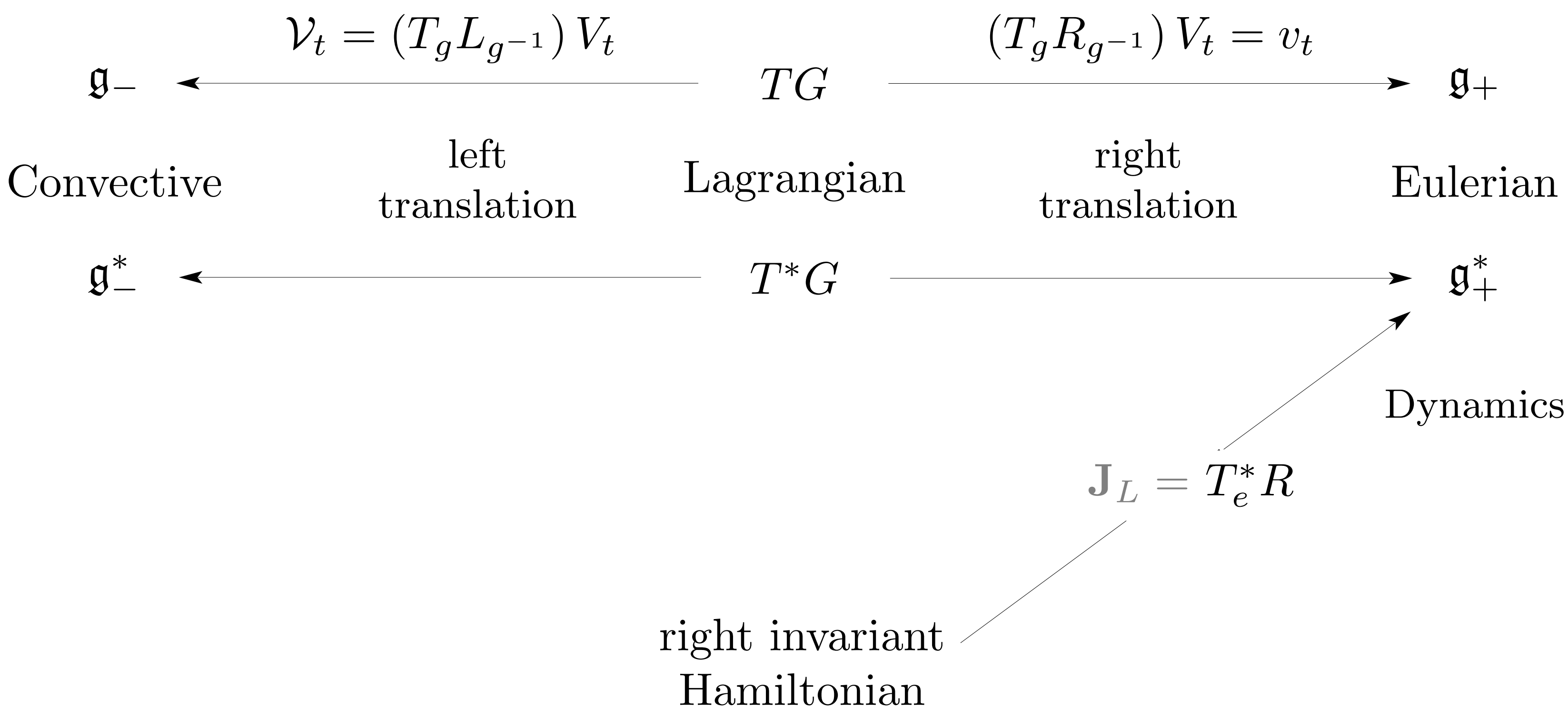
$\mathcal{H}(\ell) = \int_{T^*Q} H(q,p)\,\ell(q,p)\,d\varpi$



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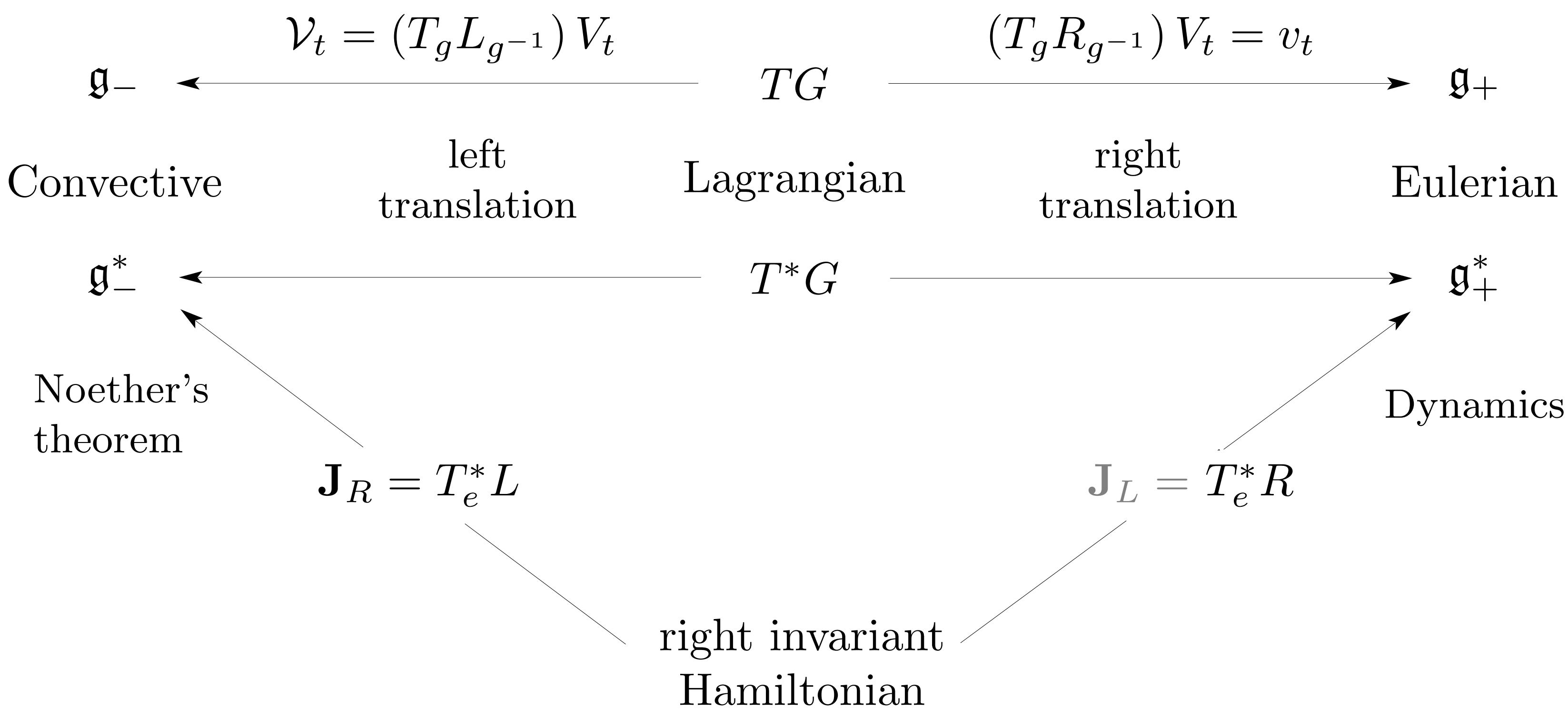
right invariant Hamiltonian



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T^*G
$\mathfrak{g}^*_+$
Noether's theorem
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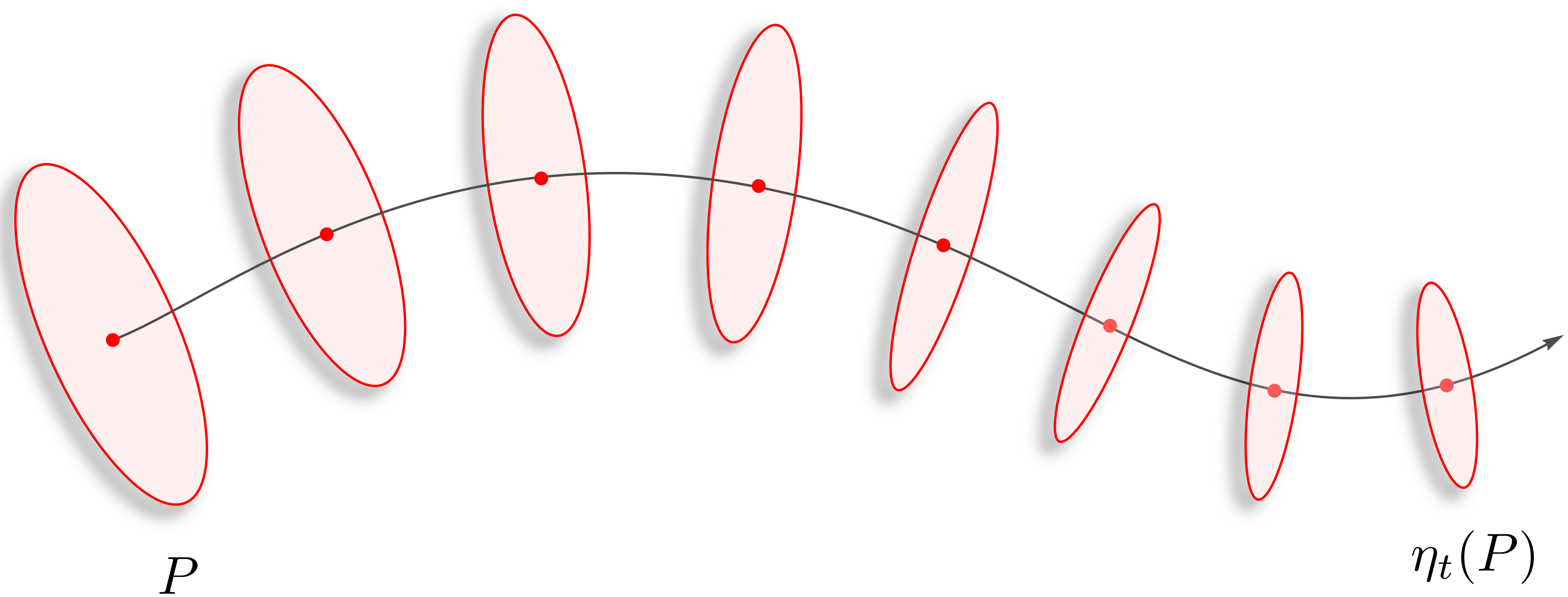
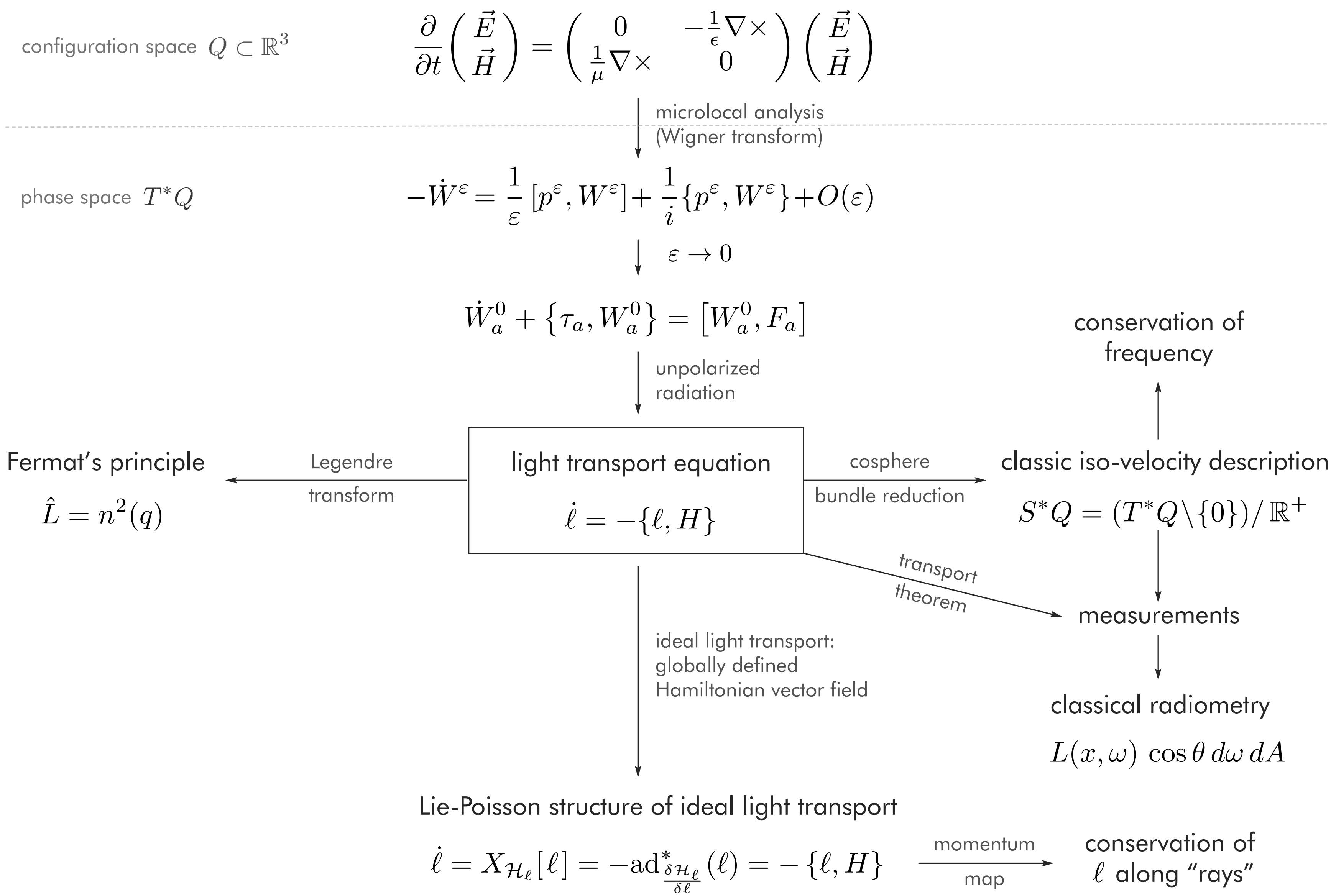
$\mathbf{J}_R = T^*_e L$
$\mathbf{J}_L = T^*_e R$
right invariant Hamiltonian


P
$\eta_t(P)$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	ideal fluid dynamics	ideal light transport
Lie group		
Lie algebra		
dual Lie algebra		
coadjoint action		
momentum map		

	ideal fluid dynamics	ideal light transport
Lie group	$\mathrm{Diff}_\mu(Q)$	$\mathrm{Diff}_{\mathrm{can}}(T^*Q)$
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	ideal fluid dynamics	ideal light transport
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momentum map		

	ideal fluid dynamics	ideal light transport
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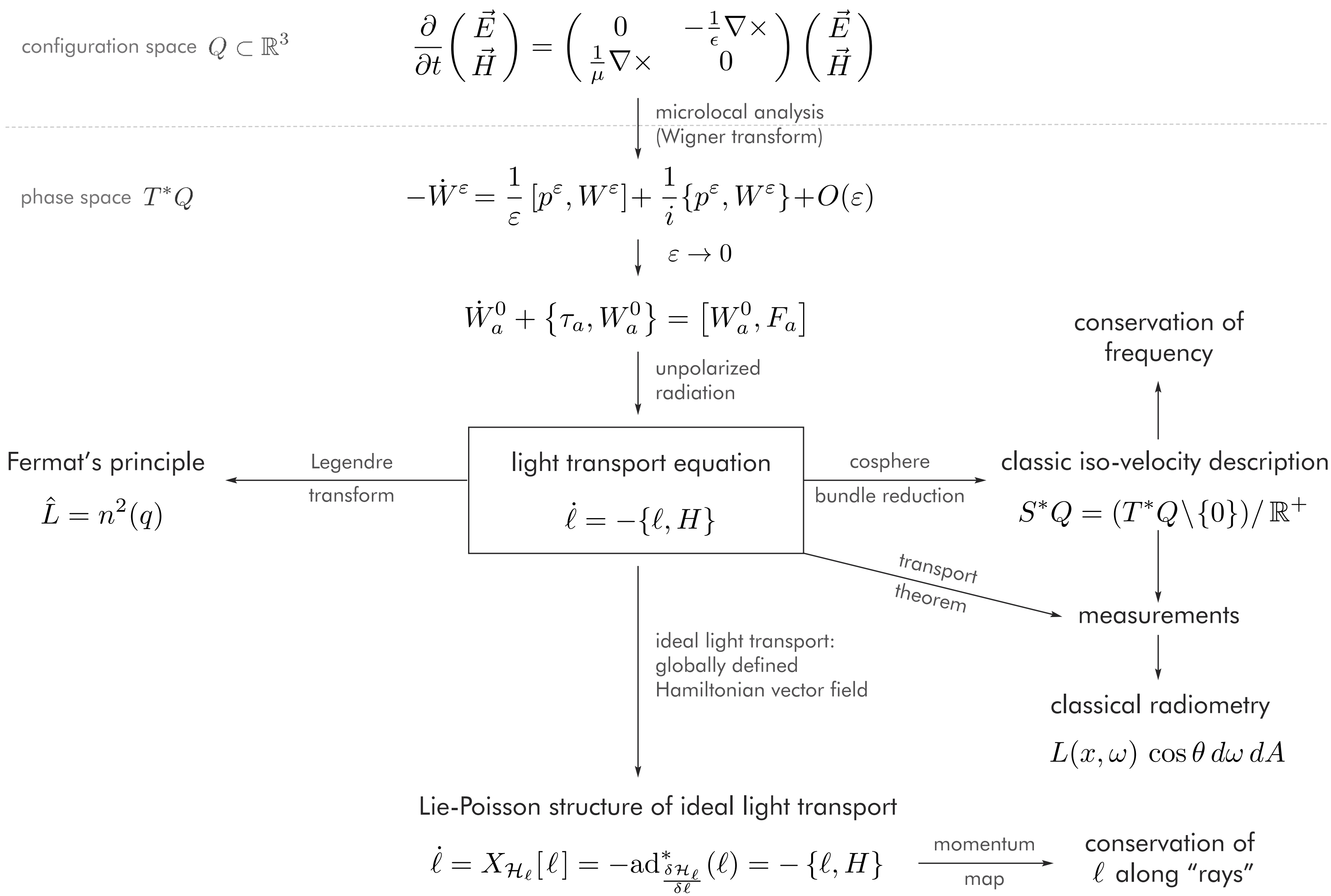
momentum map	Kelvin's circulation theorem	conservation of radiance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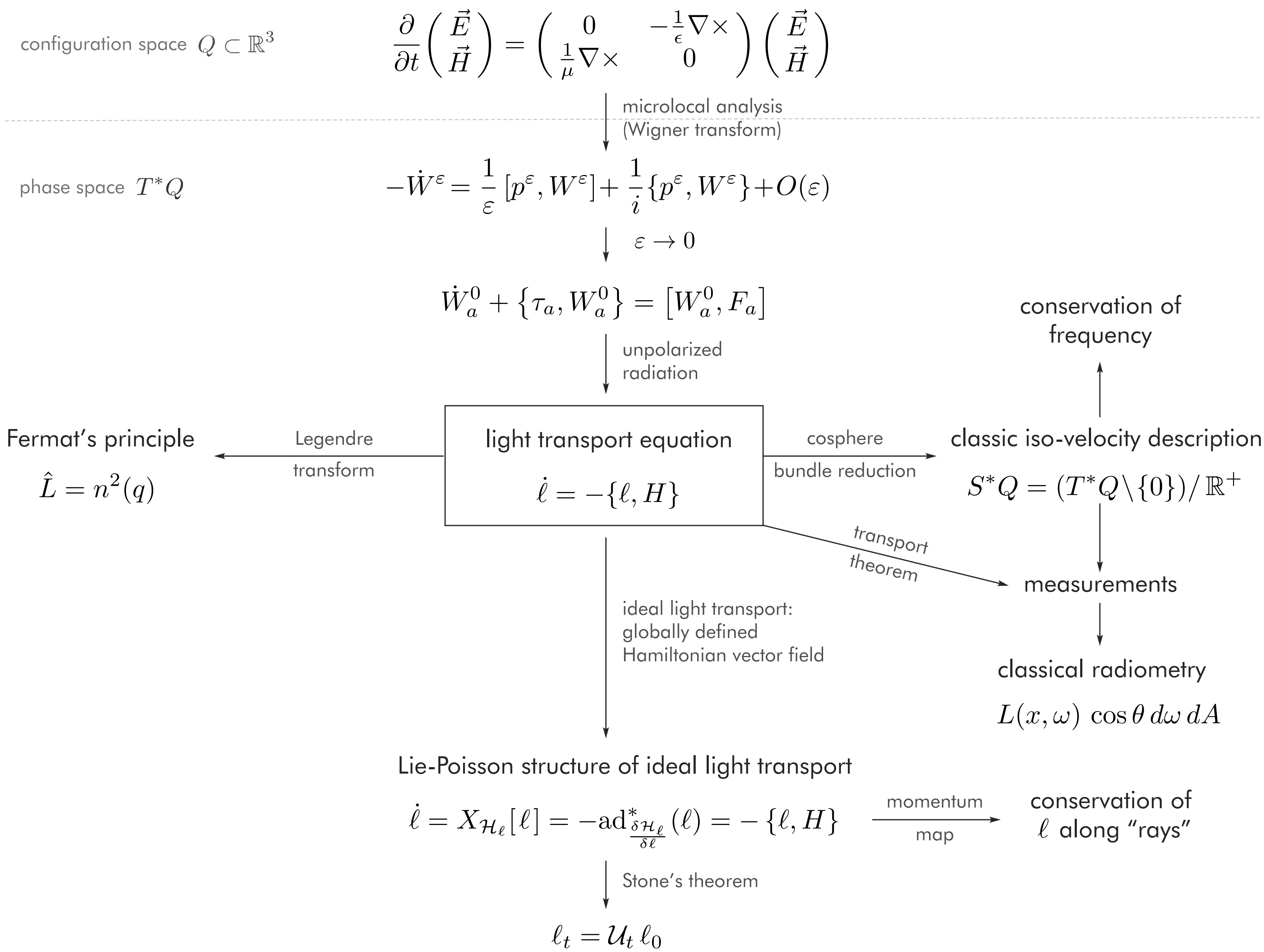
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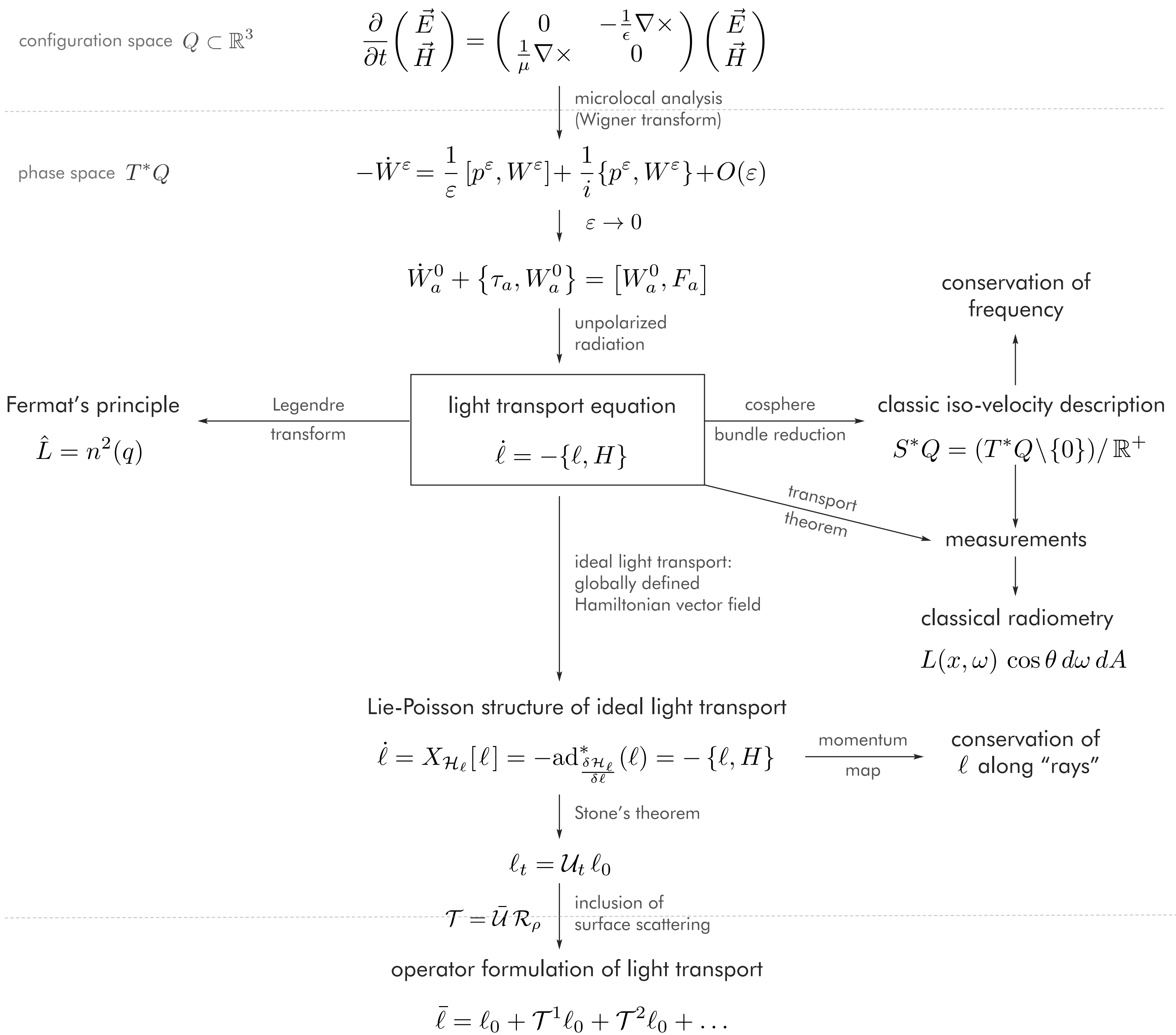
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$\mathcal{T} = \bar{\mathcal{U}} \, \mathcal{R}_\rho$
inclusion of
surface scattering
operator formulation of light transport
$\bar{\ell} = \ell_0 + \mathcal{T}^1 \ell_0 + \mathcal{T}^2 \ell_0 + \ldots$

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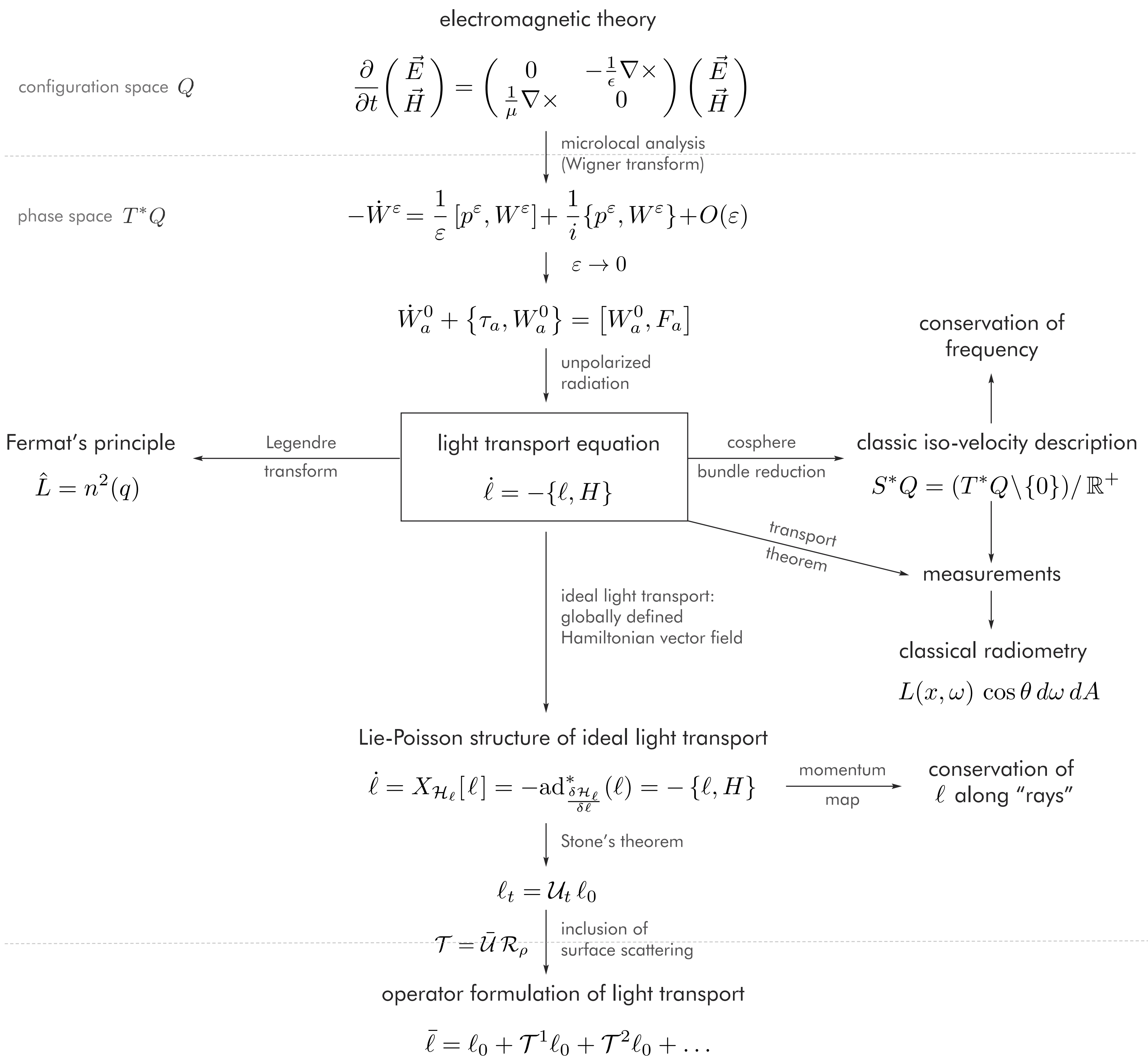
Jerry Marsden, and Boris Khesin.

lessig@dgp.toronto.edu

www.dgp.toronto.edu/~lessig/dissertation/


electromagnetic theory
configuration space Q
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