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Discretization of Hamiltonian Incompressible Fluids

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Introduction The motion of an incompressible, inviscid Euler fluid can be described geometrically by Arnold's classic Lagrangian formulation [1, 5], which says that the motion of the fluid is given by geodesics on the Lie group $\mathscr{D}_{\text{vol}}(Q)$ of volume-preserving diffeomorphisms on a domain Q. In recent years this formulation has been discretized to create a symplectic numerical method that obeys a discrete version of Kelvin's circulation theorem [2, 3]. This talk presents an extension of this discretization that employs the Hamiltonian view of incompressible inviscid fluids [4] given by the vorticity equation.

Discretization of the Lie group action Following the approach introduced by Pavlov et al. in [2], we start by discretizing the domain Q, dividing it up into N cells (our application will specifically use a Cartesian grid). We then define a set of functions (which we view as discrete 0-forms) by specifying one value per cell. These may be written as vectors of length N. The action of $\mathscr{D}_{\text{vol}}$ on Q is represented by matrices acting on these vectors. We expect to preserve the discrete L^2 inner product of functions, and leave constant functions unchanged. On a regular Cartesian grid, this is achieved using the Lie group G of N-by-N orthogonal, signed-stochastic matrices.

The velocity of an incompressible fluid may be viewed as an element of the Lie algebra $\chi_{\text{vol}}(Q)$ of divergence-free vector fields which corresponds to the Lie group $\mathscr{D}_{\text{vol}}(Q)$. Similarly, the velocity in our discretized picture may be given as an element of the Lie algebra $\mathfrak{g}$ of row-null antisymmetric matrices that corresponds to our Lie group G. Discrete 1-forms and 2-forms are defined via antisymmetrizations of our 0-forms and 1-forms, respectively. It is shown in [3] that we can view $\mathfrak{g}^*$ as the space of discrete 1-forms modulo exact 1-forms, in direct analogy with a similar construction for the continuous dual Lie algebra χ^*_{vol}.

If we use the entire Lie algebra $\mathfrak{g}$, this will allow interactions between cells that are far away from each other. Such an interaction is both physically implausible over short timesteps and also computationally expensive to calculate. Accordingly, we introduce a constraint such that, for $A \in \mathfrak{g}$, an element A_{ij} is zero unless cells i and j are neighbors. This constraint is non-holonomic, since, in general, it is not satisfied by the Lie bracket of two elements satisfying the constraint.

The Hamiltonian (vorticity) equations It is shown in [4] that the equations for an incompressible, inviscid fluid on χ^*_{vol} can be derived from the Lie-Poisson equations $\dot{F} = \{F, H\}$ using a Hamiltonian given by

$$H = \frac{1}{2}\int \langle \Delta^{-1}\omega, \omega\rangle \,\mathrm{d}x \tag{1}$$

and the Lie-Poisson bracket $\{\,,\}: \mathscr{F}(\chi^*_{\text{vol}}) \times \mathscr{F}(\chi^*_{\text{vol}}) \to \mathscr{F}(\chi^*_{\text{vol}})$ which is given by

$$\{F, G\}(\omega) = \int \langle \omega, \left[\frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega}\right]\rangle \mathrm{d}x \tag{2}$$

where the square brackets denote the Lie bracket on χ_{vol}, which is just the usual Jacobi-Lie bracket of vector fields [5].

We can construct a similar derivation on our finite-dimensional dual Lie algebra $\mathfrak{g}^*$, using the Lie bracket on $\mathfrak{g}$ to define a Lie-Poisson bracket that takes the exact same form. To define a discrete Hamiltonian, we will need a Laplacian. We use a discrete version of the Hodge Laplacian, which is written as $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$, where d is the discrete exterior derivative [6], and δ is the associated codifferential, defined with respect to the Frobenius inner products for our discrete 1-forms and 2-forms.

A set of manipulations precisely analogous to those in [4] for the continuous case will then yield semidiscretized vorticity equations of the form:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} + \mathrm{ad}^*_v \omega = 0 \text{ with } v = \delta\Delta^{-1}\omega \tag{3}$$

We can use a variational Lie-Poisson integrator such as those outlined in [7] to approximate the time derivative, and the result is a fully-formed numerical method for inviscid, incompressible fluids, based on vorticity.

Results Numerical experiments confirm that the method described above has several desirable properties. It displays the approximate long-time energy conservation that is expected from a symplectic method. It is also unusually good at resolving details of the dynamical behavior, such as the bifurcation point created by two co-rotating vortices which may either collide or drift away from each other over time, depending on how close together they start. Where a standard method might require a 256 by 256 grid to resolve the correct behavior, for our method 80 by 80 is sufficient.

Conclusion We have presented an approach to creating numerical methods for fluids, characterized by approximating the Lie group of volume-preserving diffeomorphisms by a finite-dimensional Lie-group and then localizing the resulting equations by means of a non-holonomic constraint. This approach successfully mirrors the structure of the original problem by using a derivation that reflects the continuous picture. Numerical experiments show promising behavior such as long-time approximate energy conservation and correct dynamical behavior at low resolutions.

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