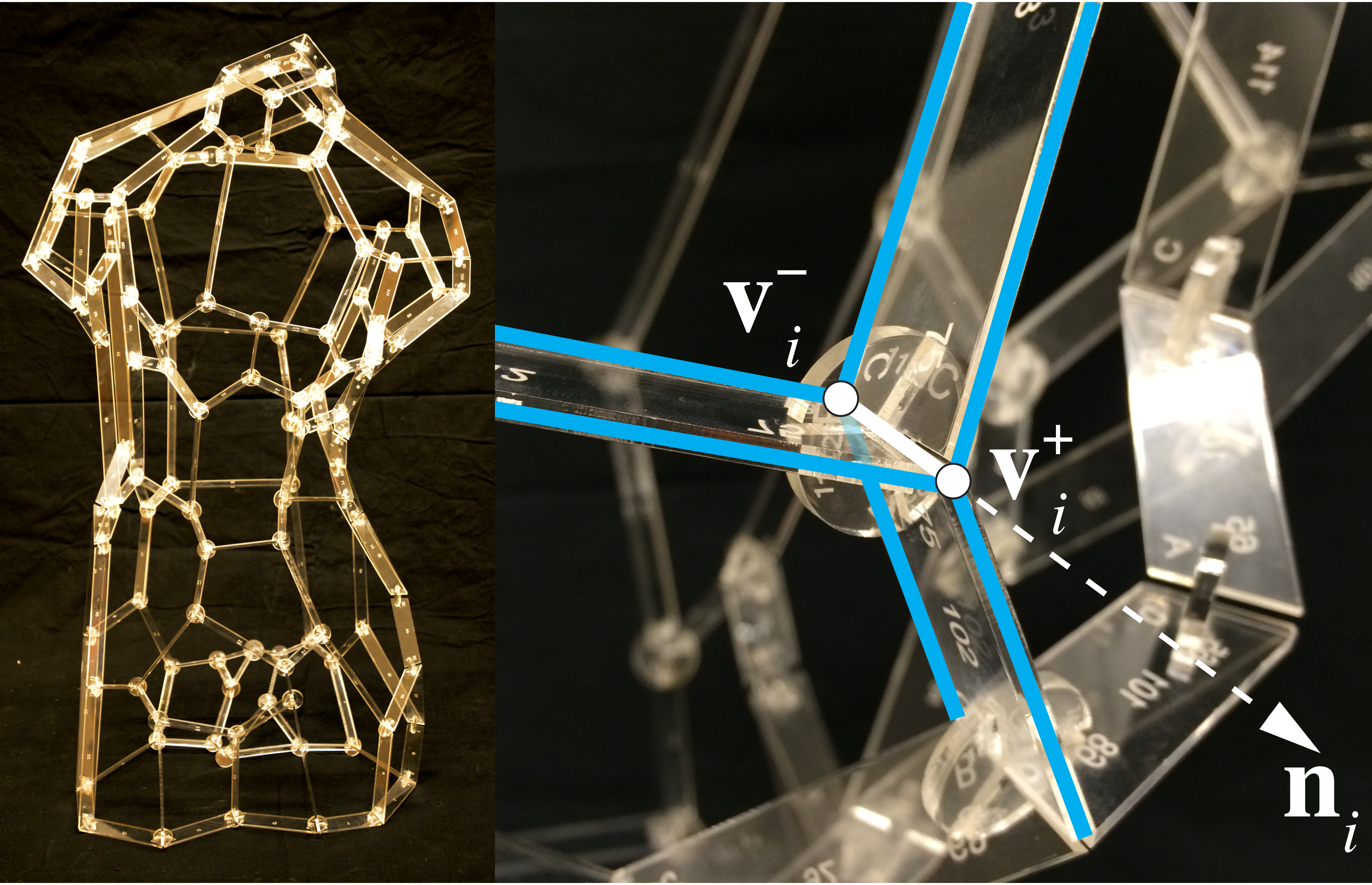
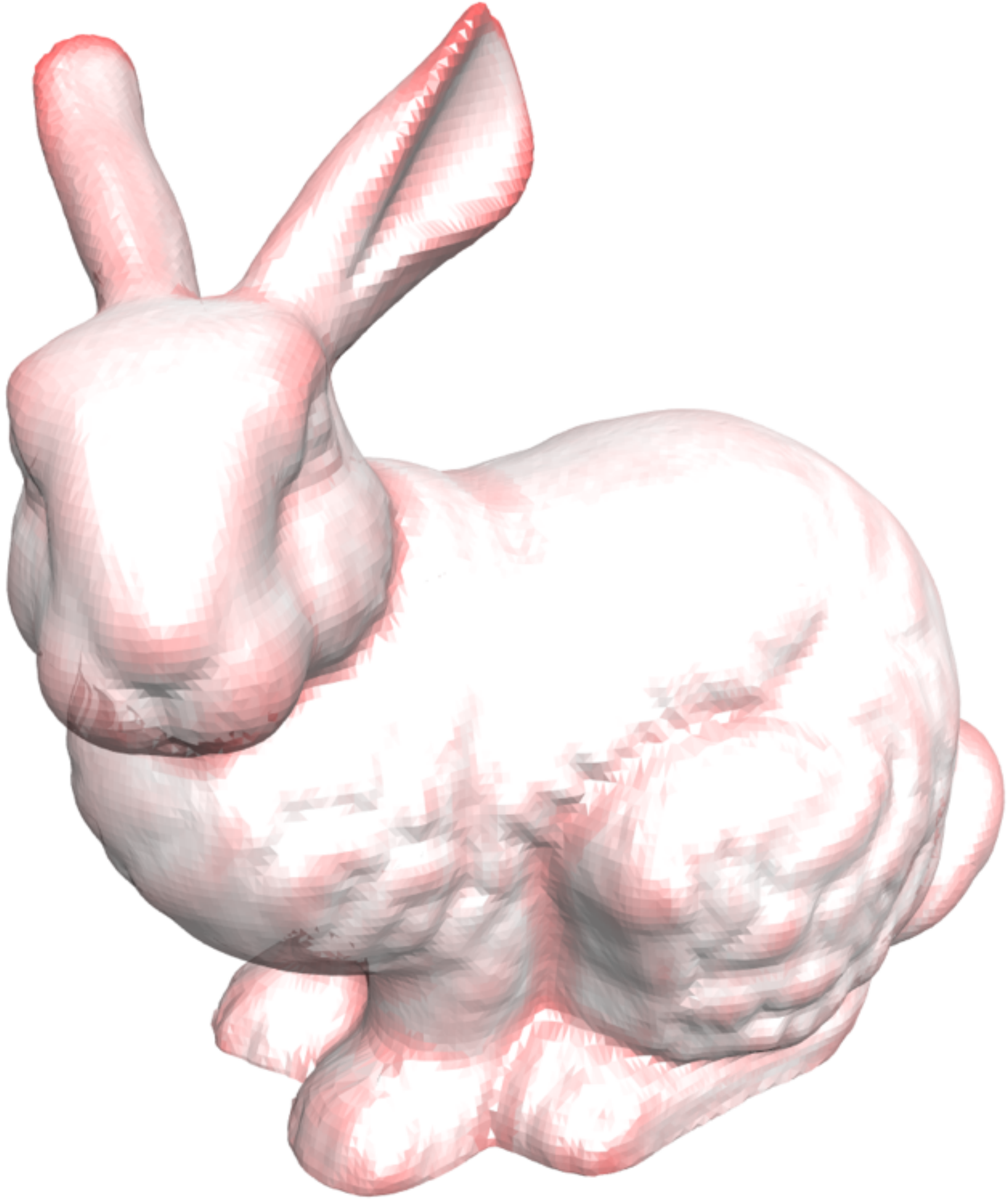
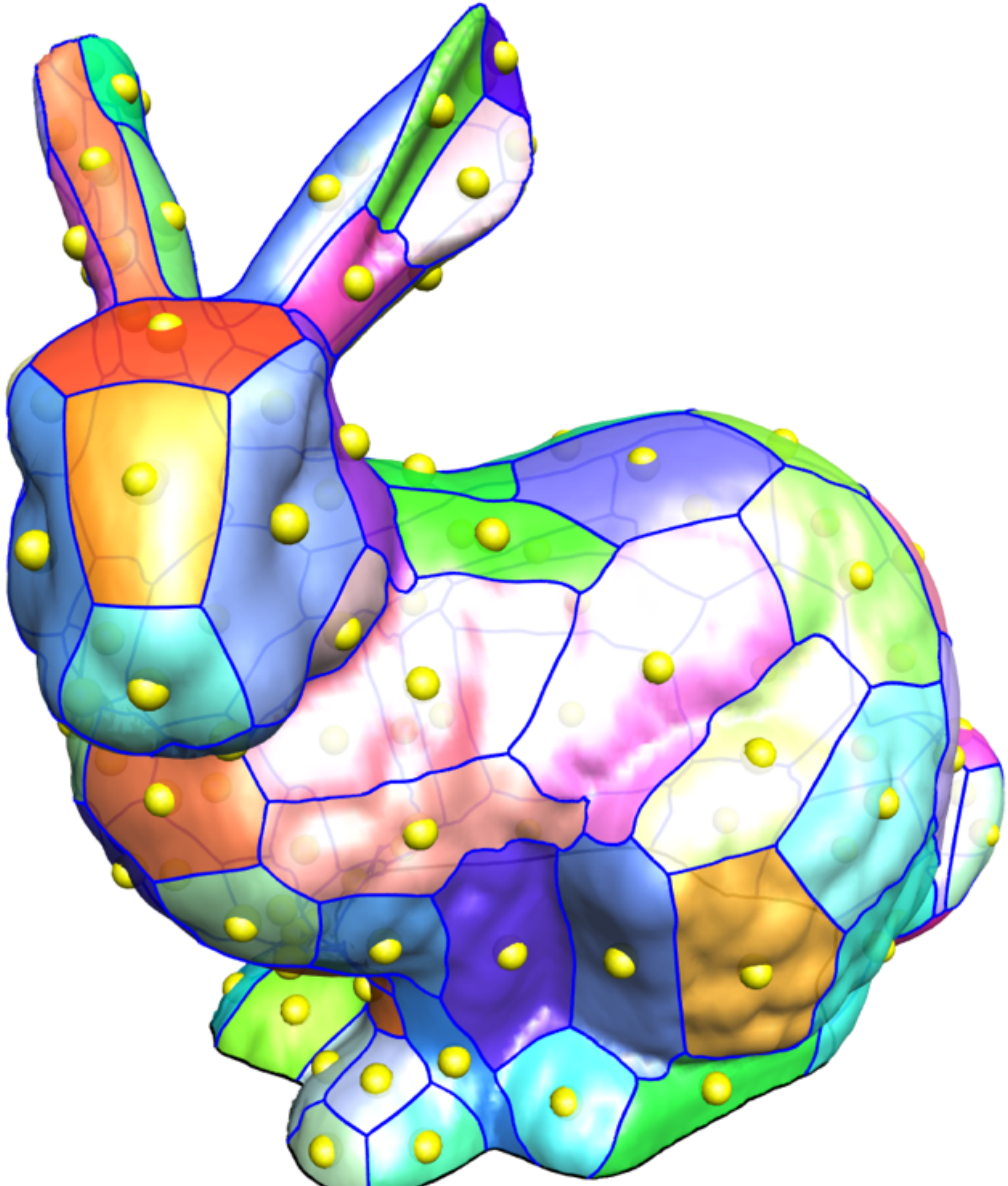
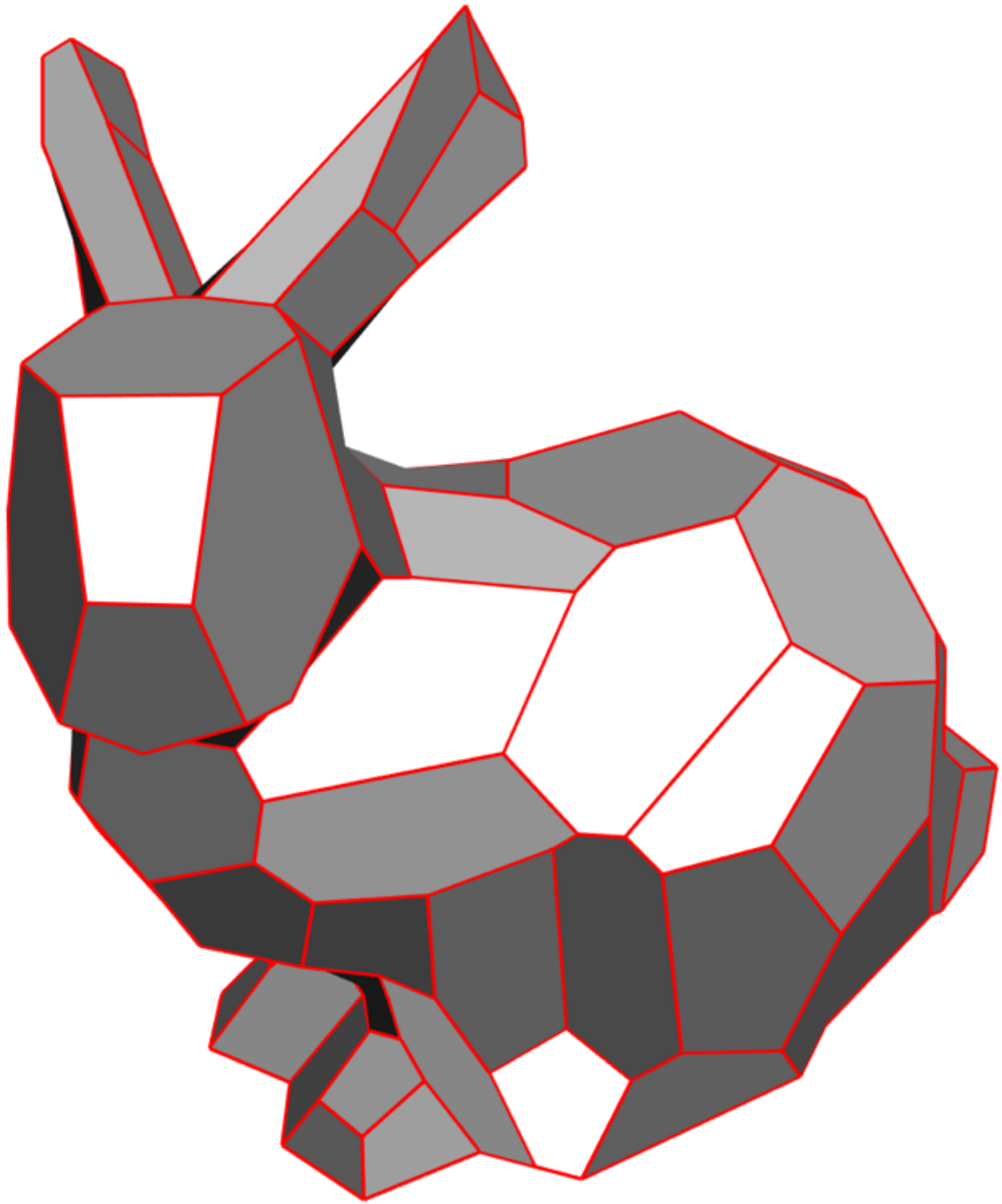
Beam Meshes

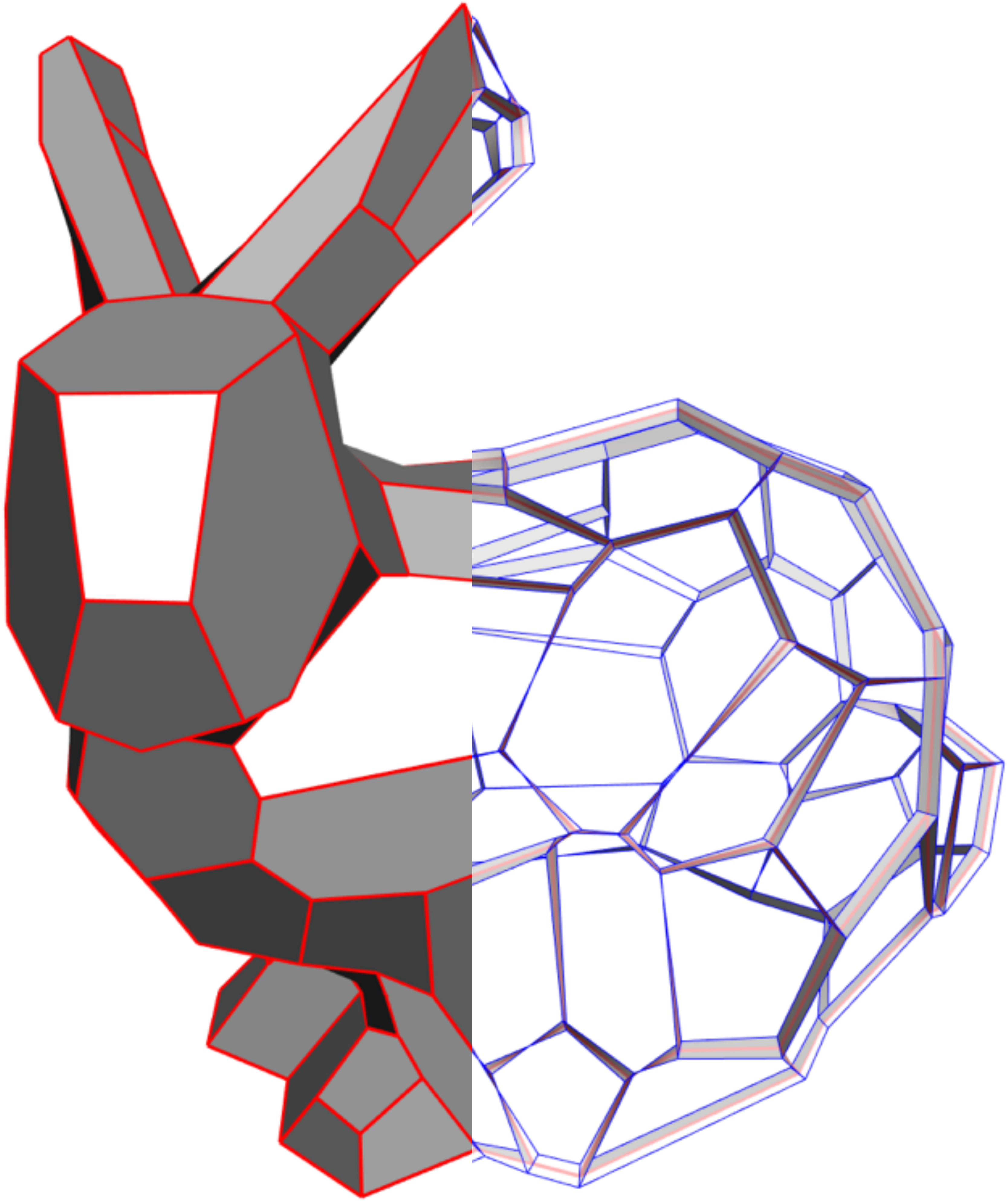
Marc Alexa

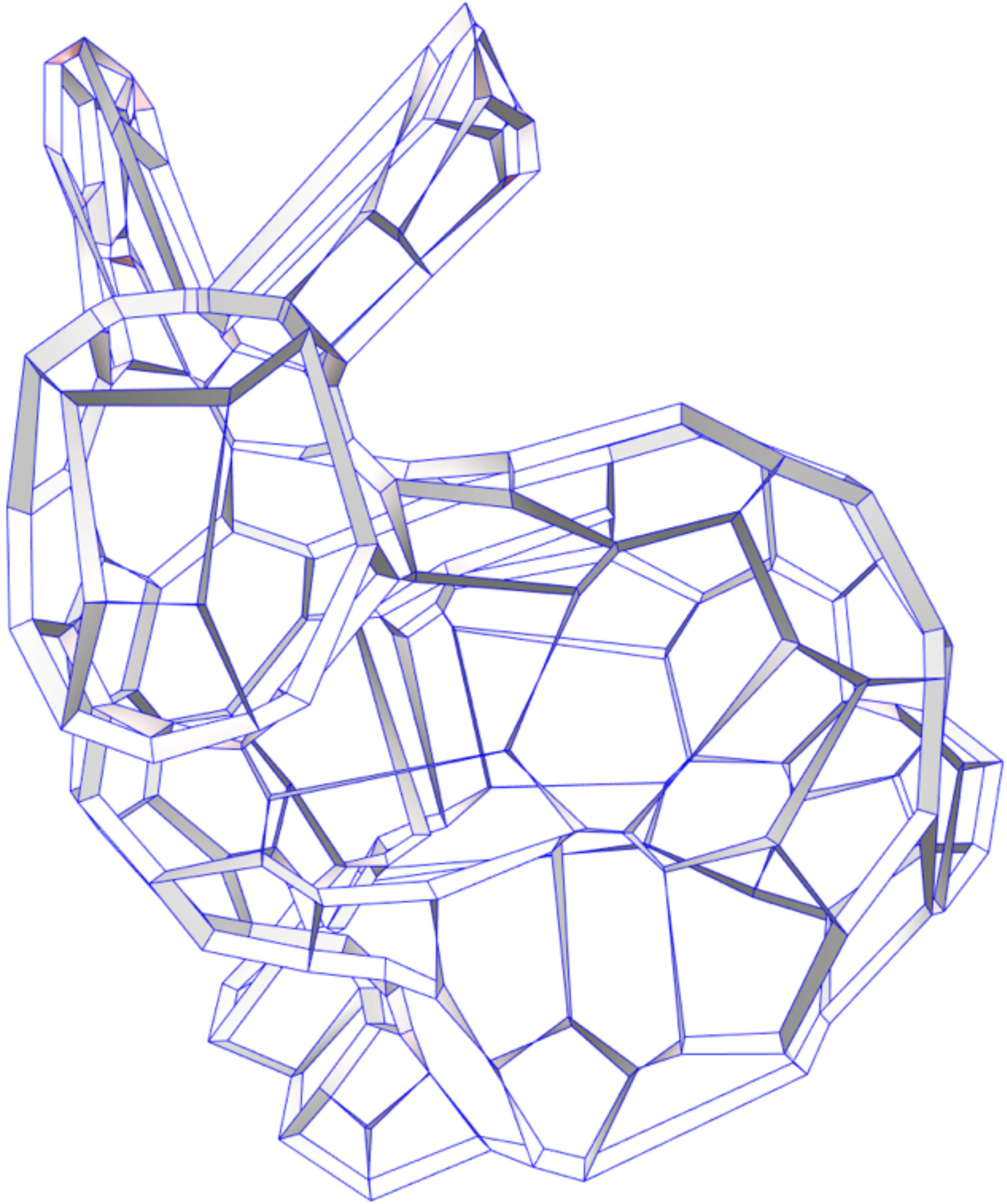

$\mathbf{v}_i^-$
$\mathbf{v}_i^+$
$\mathbf{n}_i$

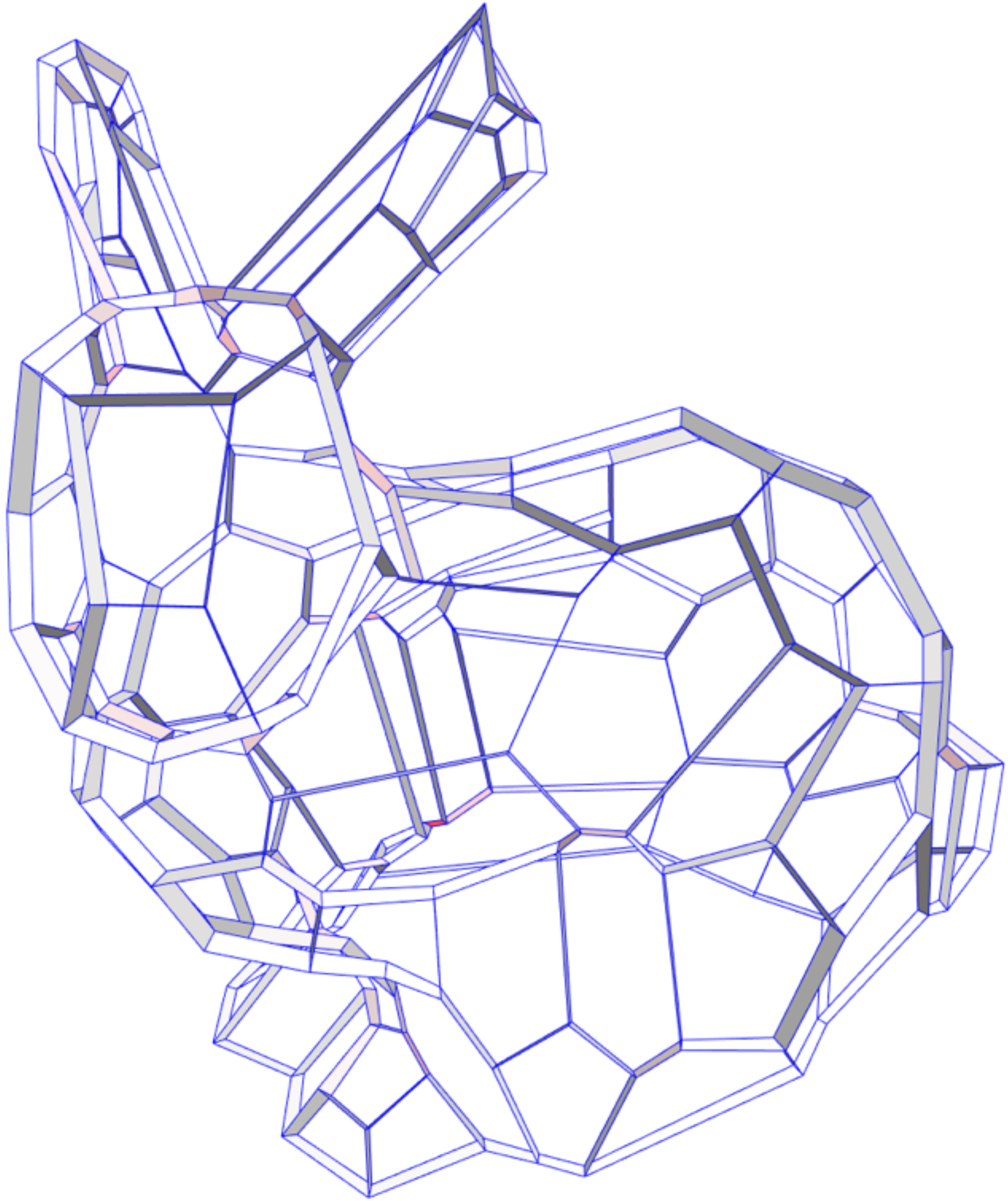


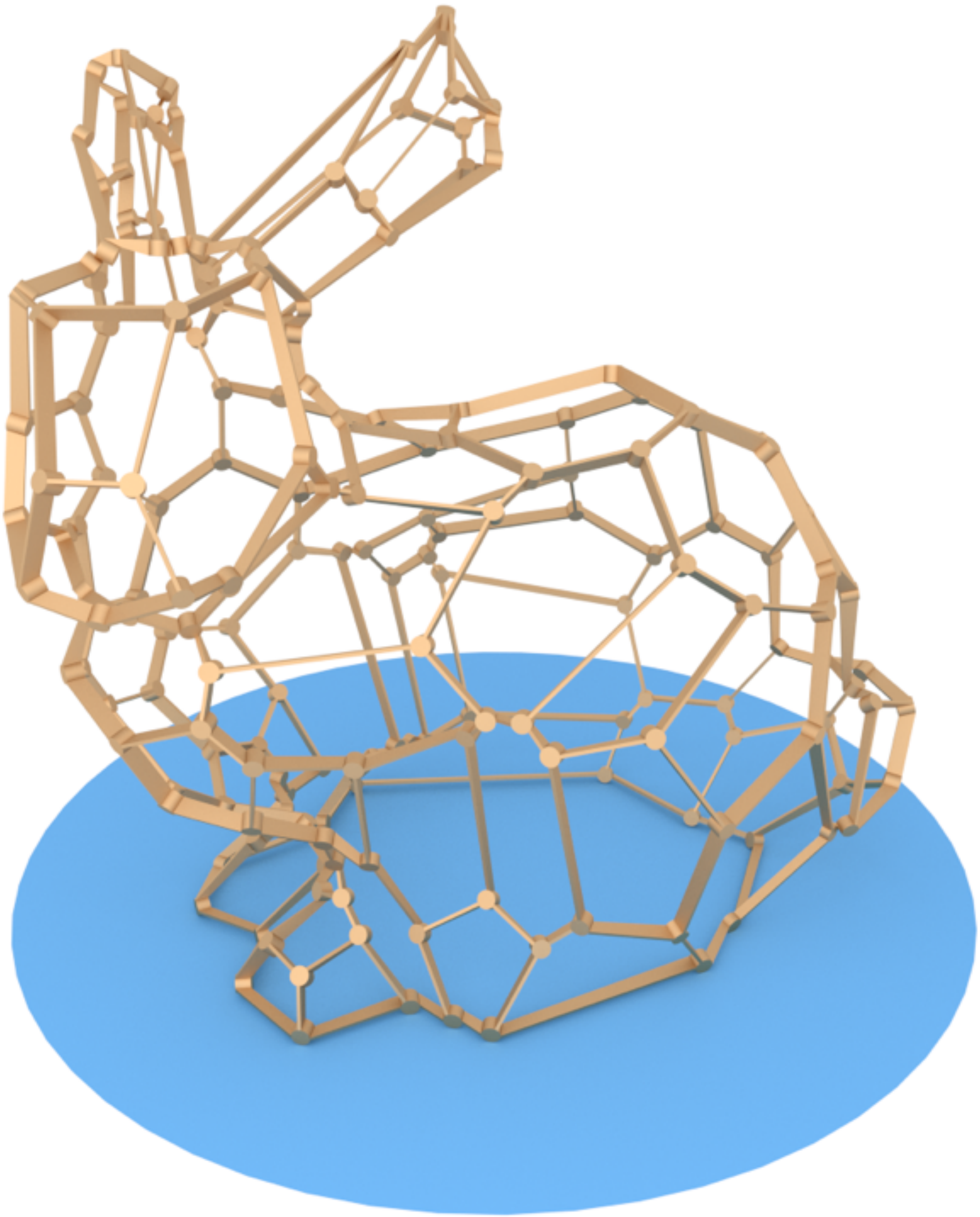


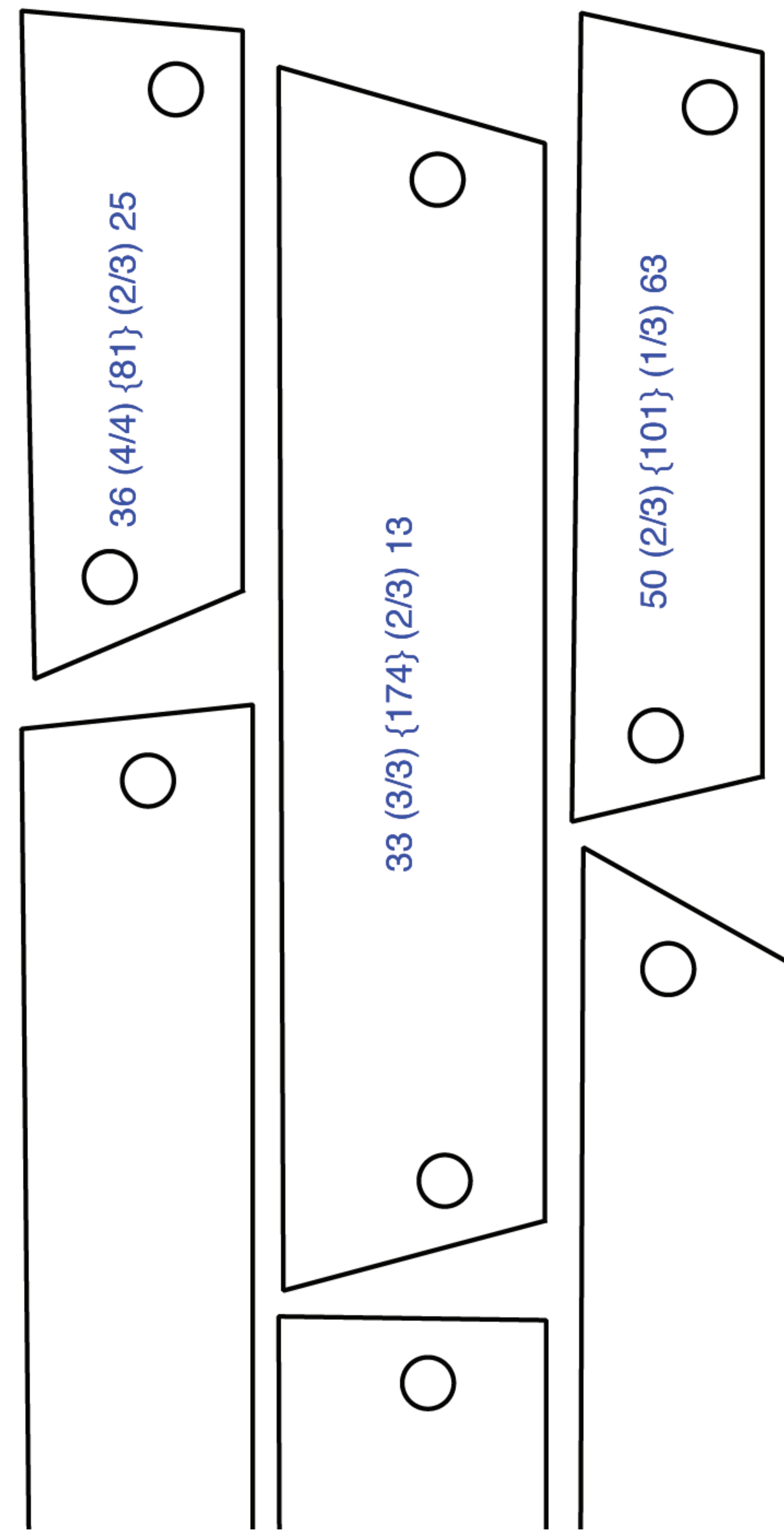


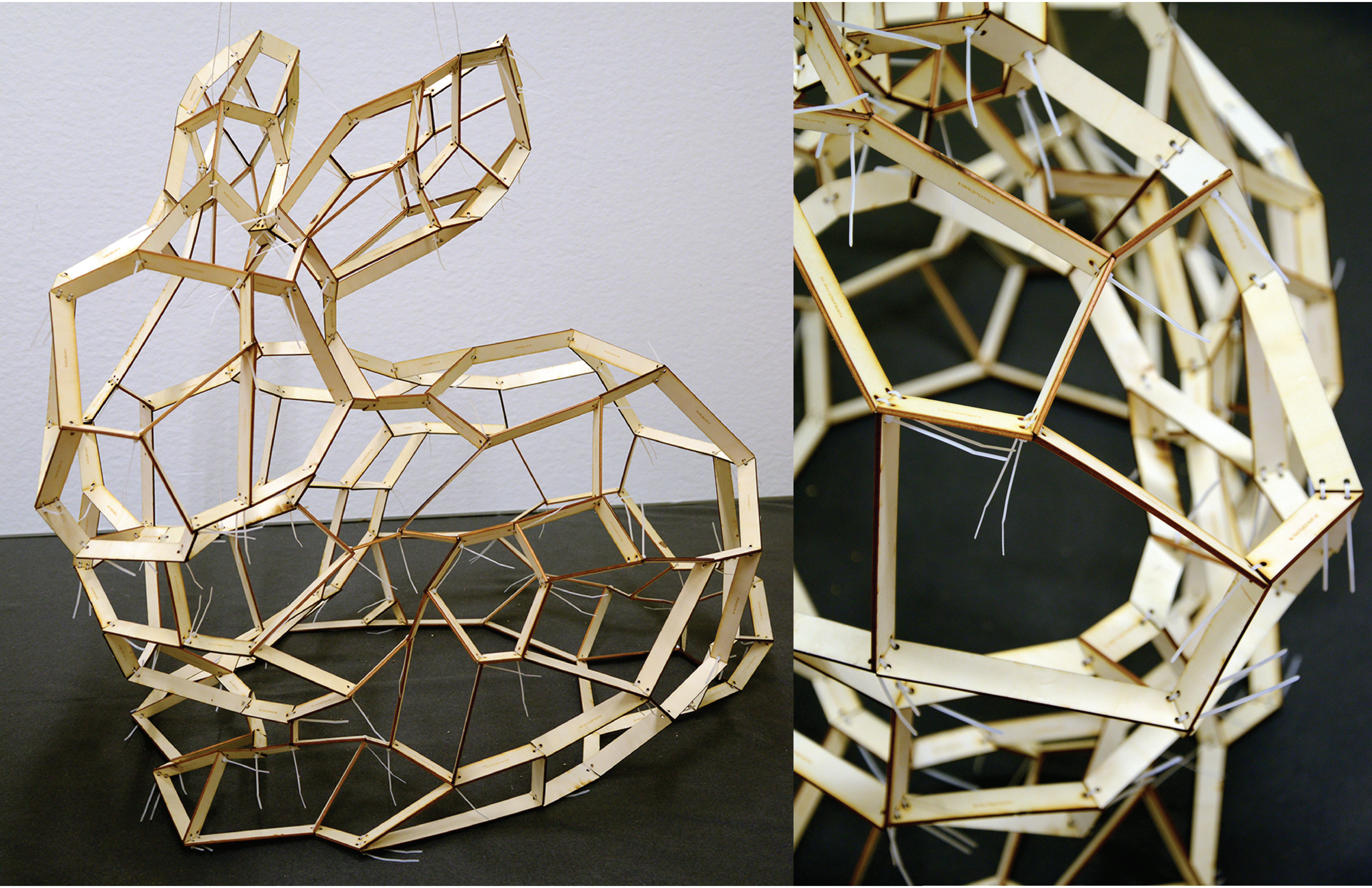




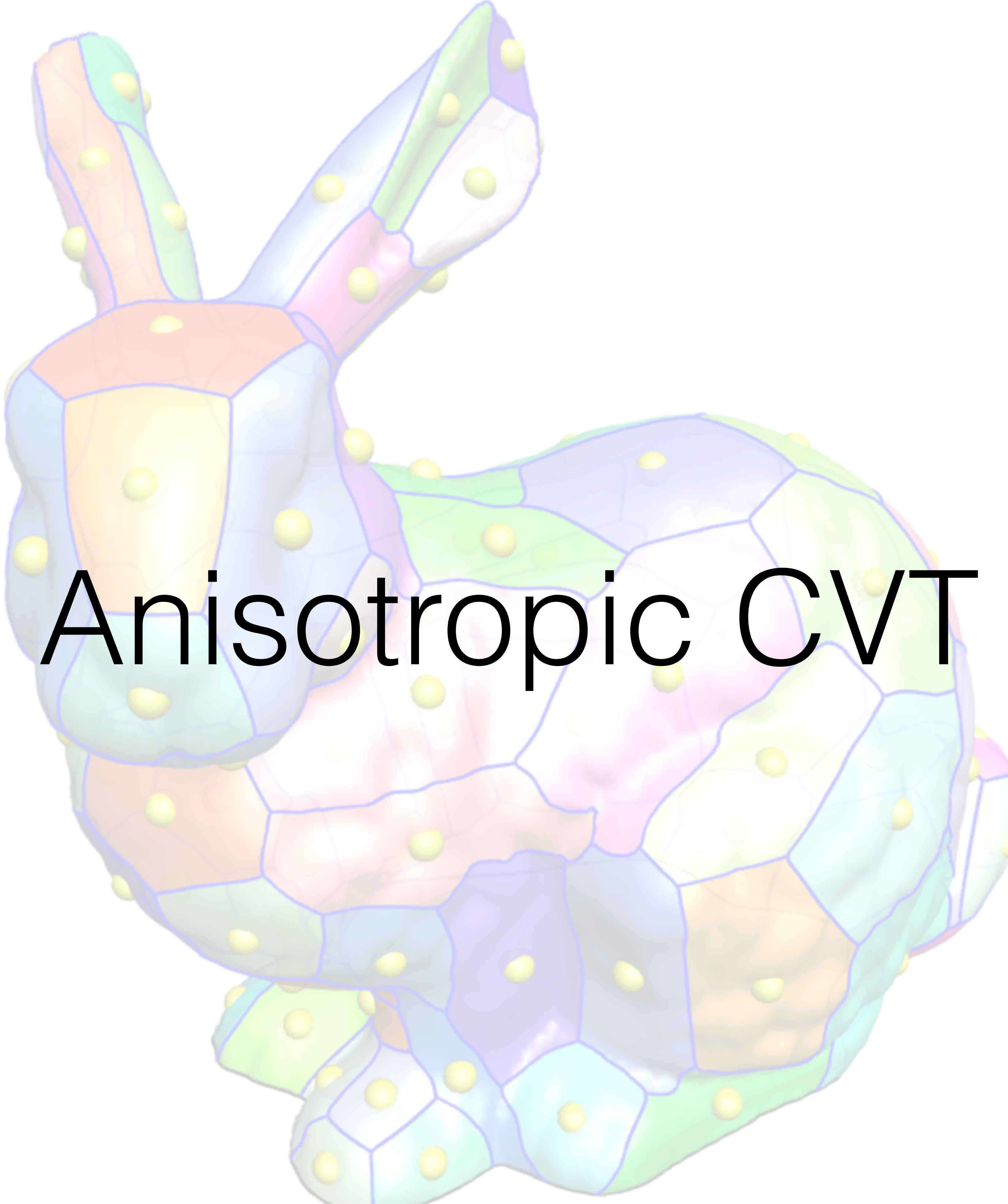


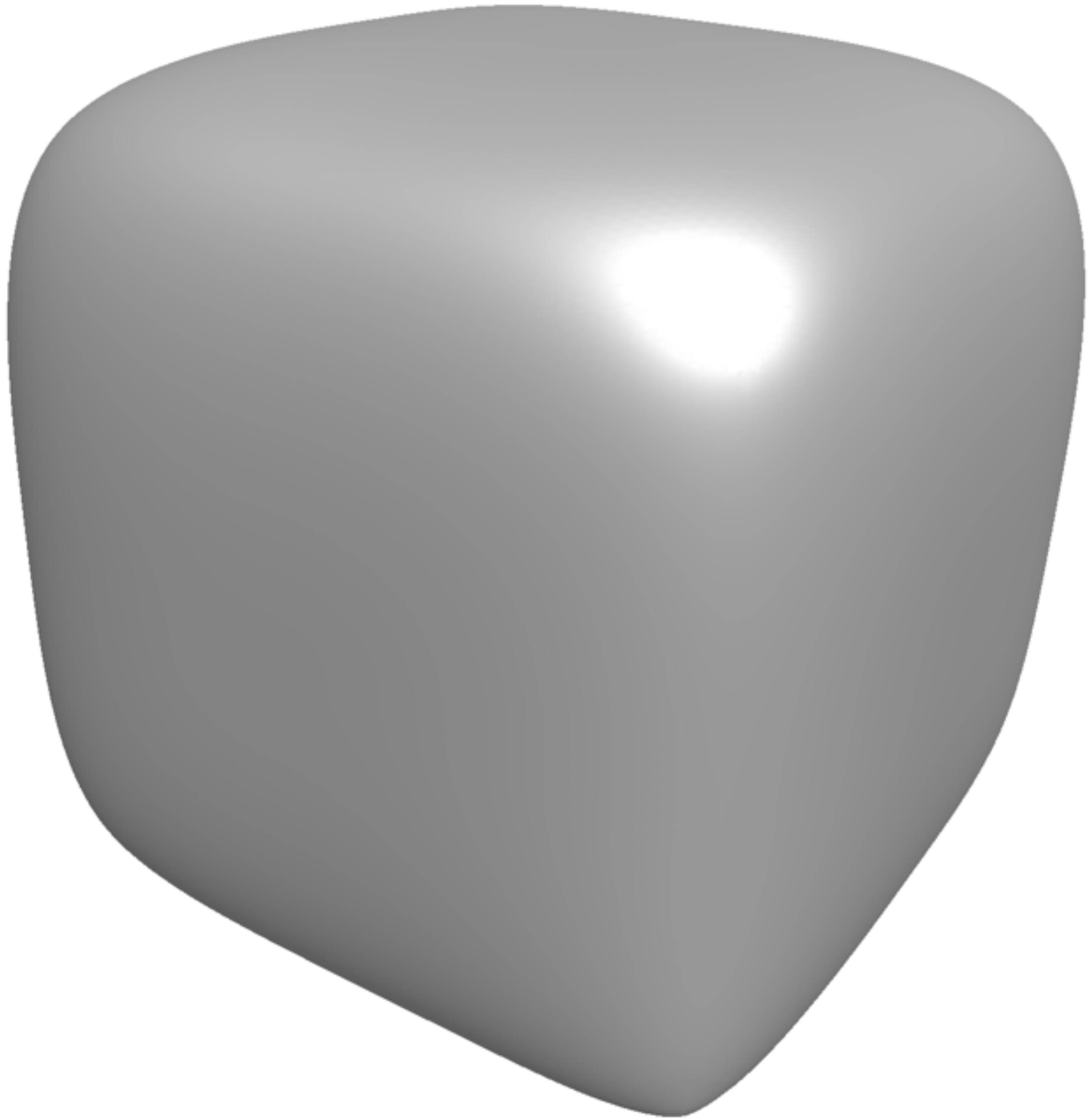


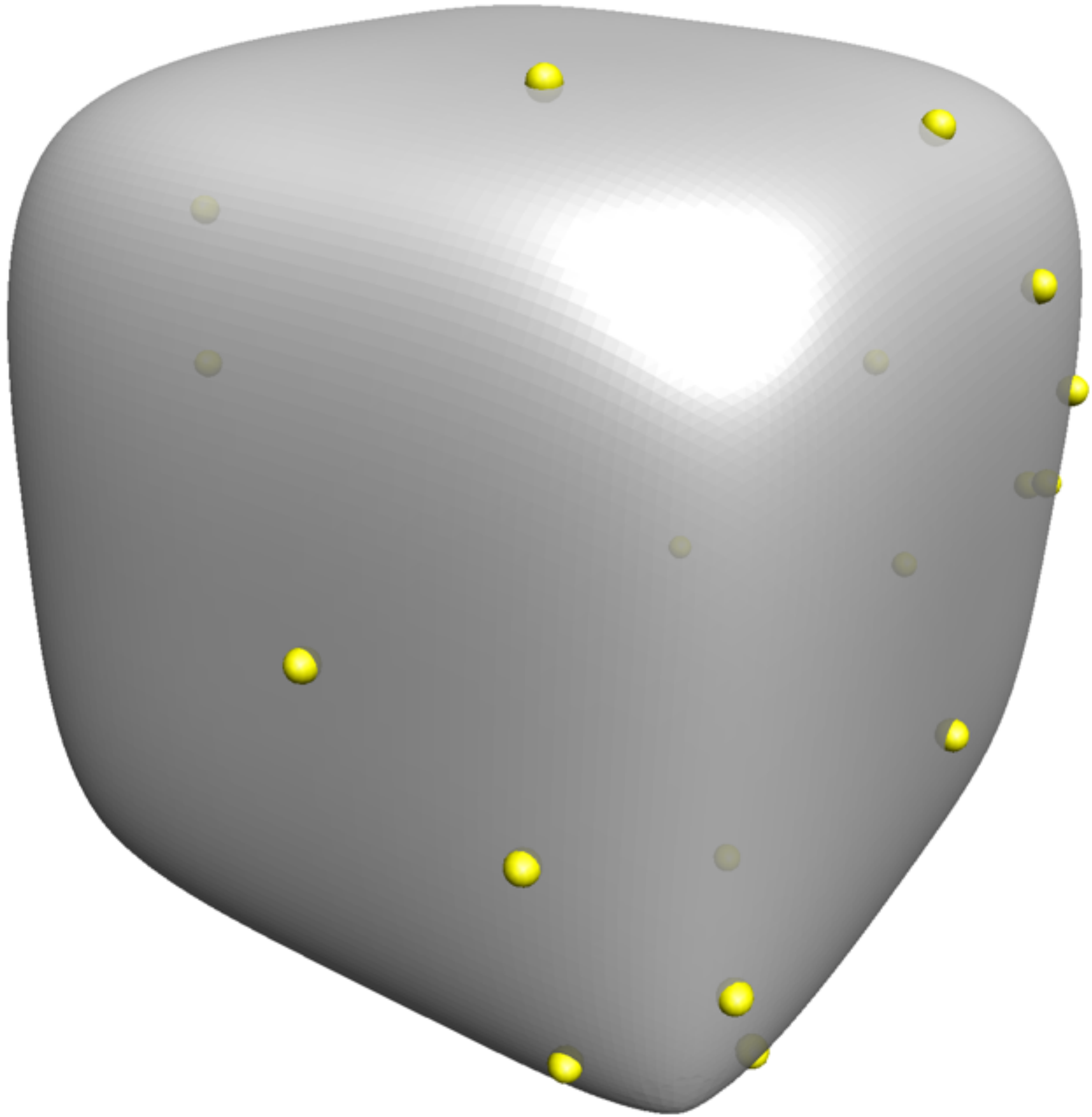

36 (4/4) {81} (2/3) 25
33 (3/3) {174} (2/3) 13
50 (2/3) {101} (1/3) 63

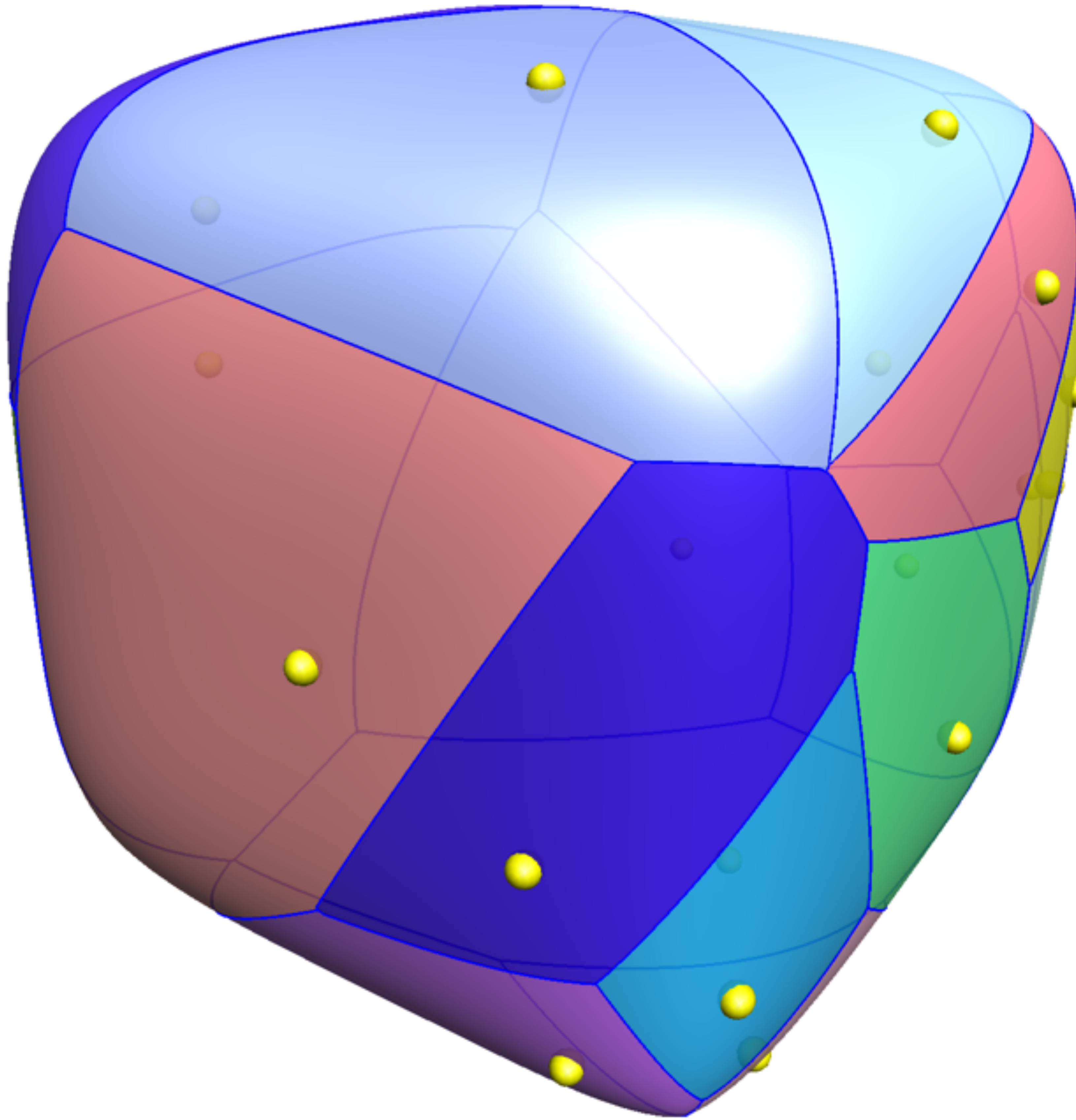


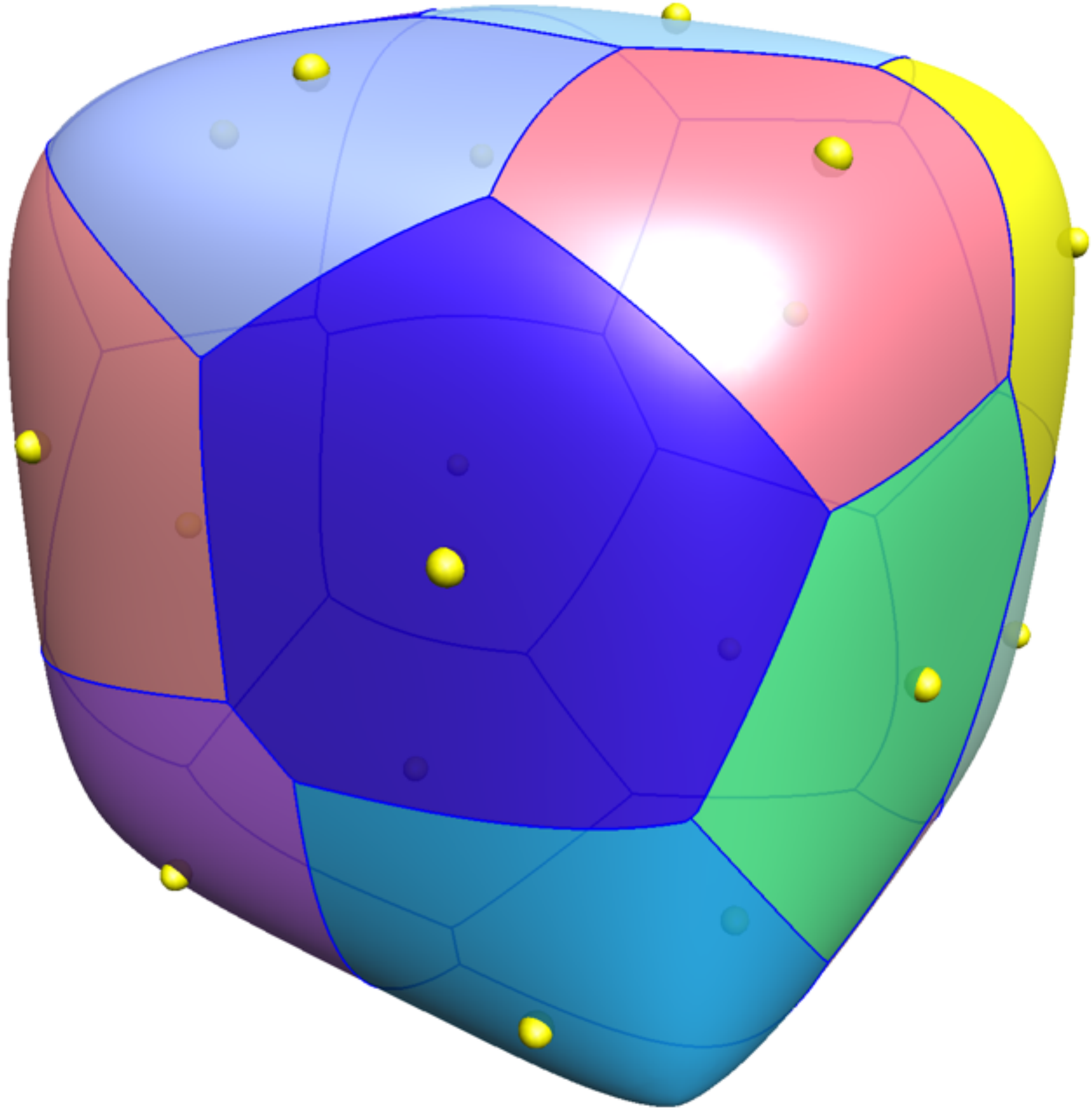
Anisotropic CVT

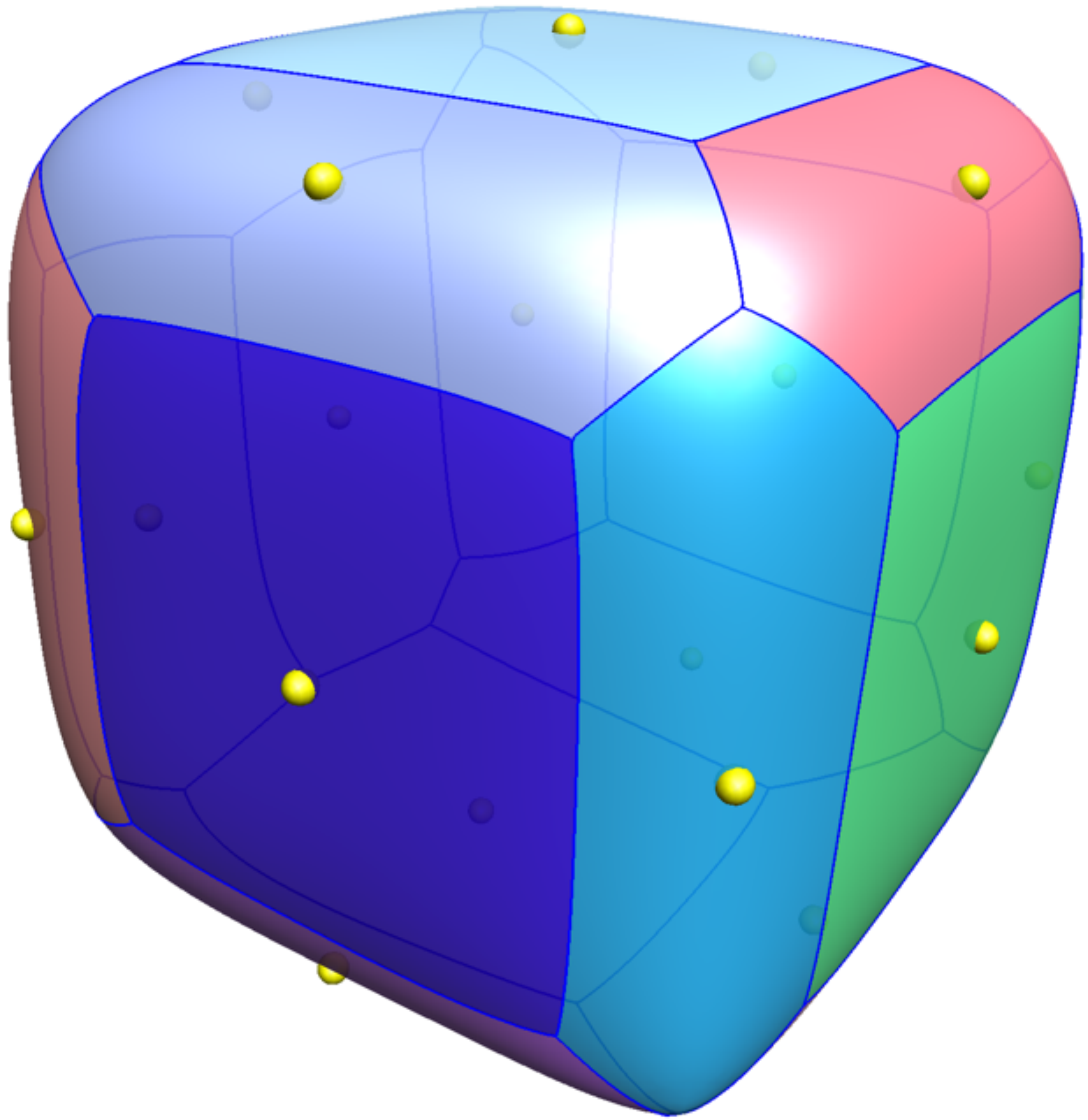


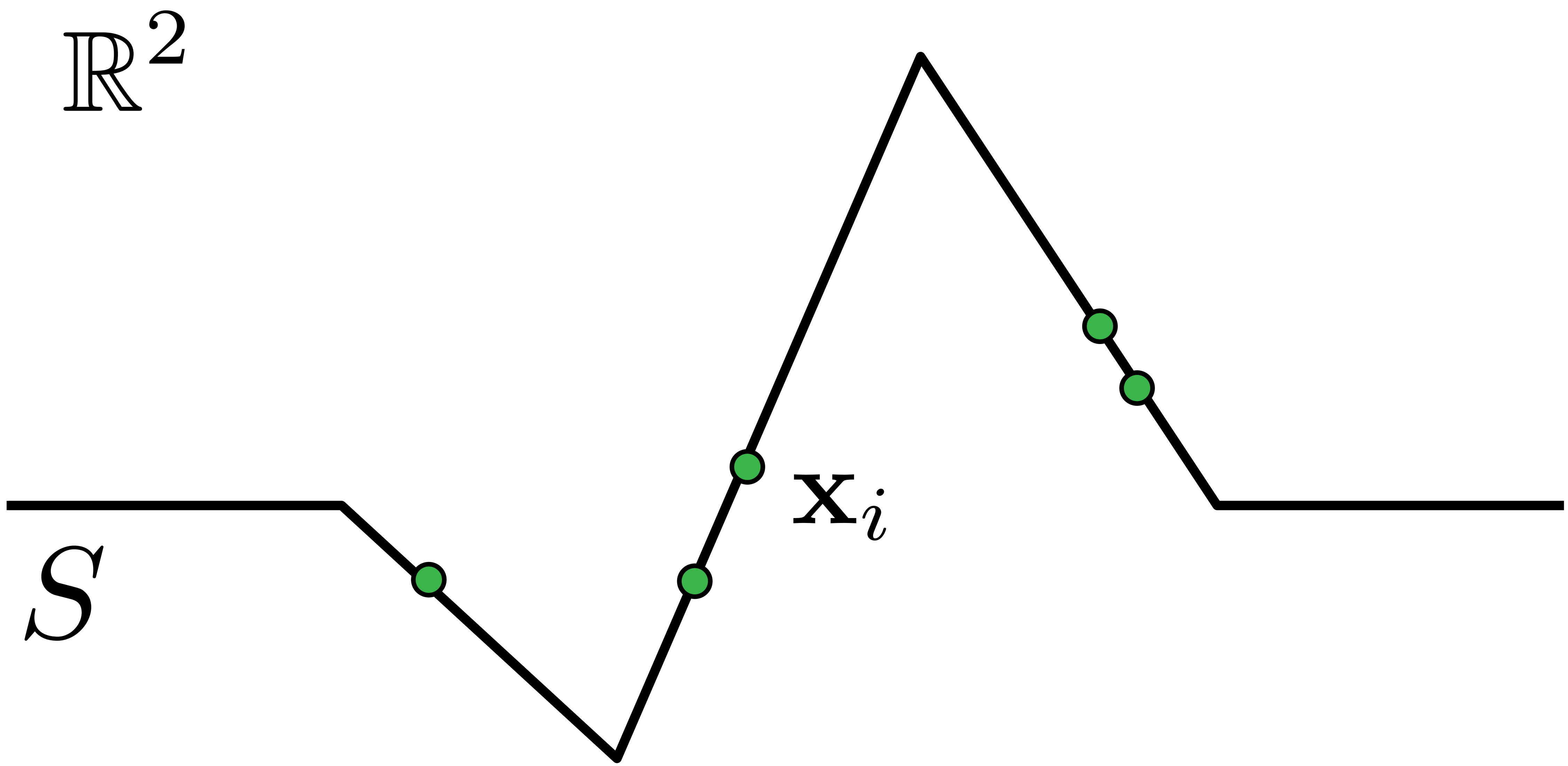


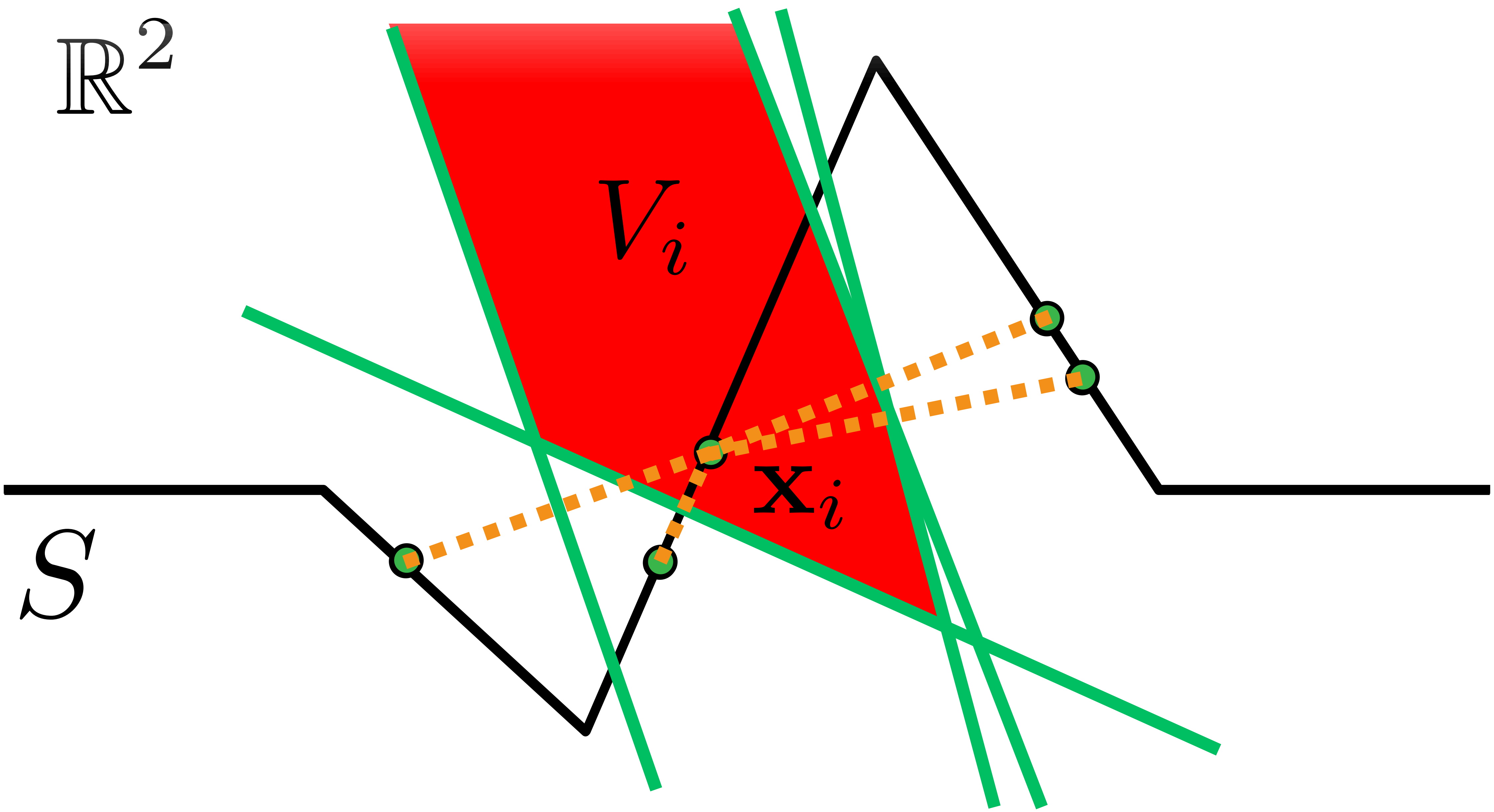




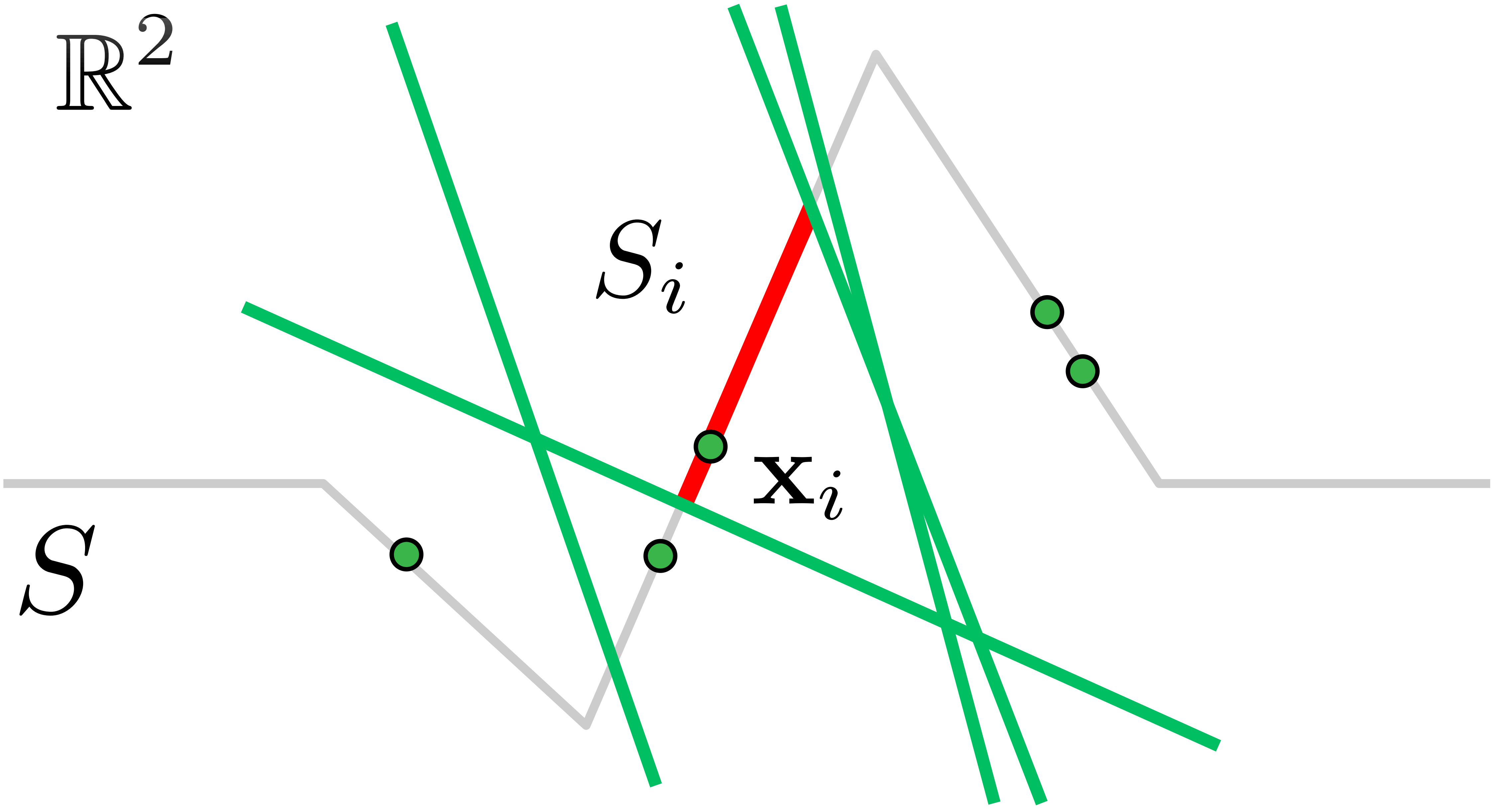





$\mathbb{R}^2$
$\mathbf{x}_i$
S



$$V_i = \{\mathbf{p} : d(\mathbf{x}_i, \mathbf{p}) < d(\mathbf{x}_j, \mathbf{p}), \forall j \neq i\} \quad \mathbf{p} \in \mathbb{R}^2$$



$$S_i = \{\mathbf{p} : d(\mathbf{x}_i, \mathbf{p}) < d(\mathbf{x}_j, \mathbf{p}), \forall j \neq i\} \quad \mathbf{p} \in S$$

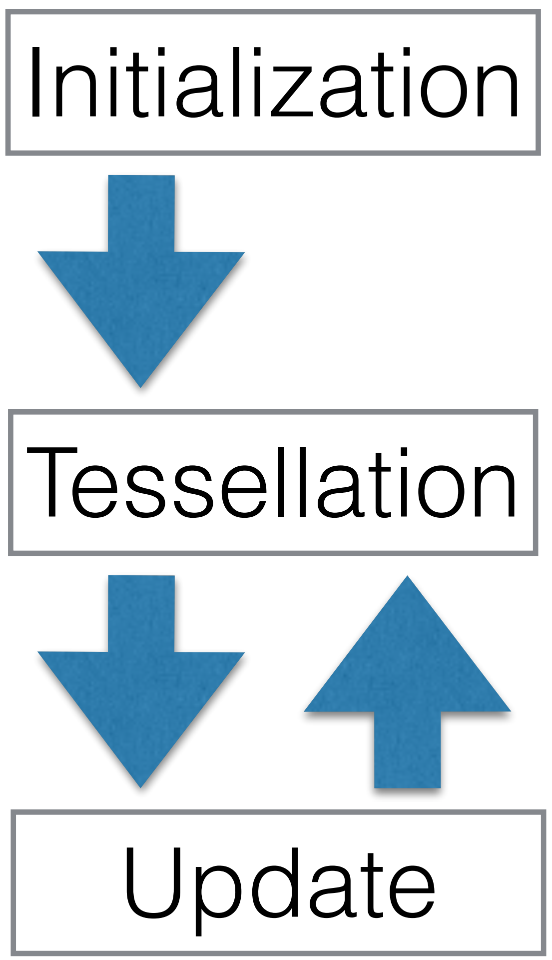
CVT

$$F(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}) = \sum_i \int_{S_i} d^2(\mathbf{x}_i, \mathbf{p}) d\mathbf{p}$$

$$\frac{\partial F}{\partial \mathbf{x}_i} = \int_{S_i} 2(\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p}$$

$$\mathbf{x}_i^{new} = \frac{\int_{S_i} \mathbf{p}\, d\mathbf{p}}{\int_{S_i} 1\, d\mathbf{p}}$$

Algorithm (Lloyd)



$\mathbf{x}_i = rand(S)$

FIXED $\mathbf{x}_i$

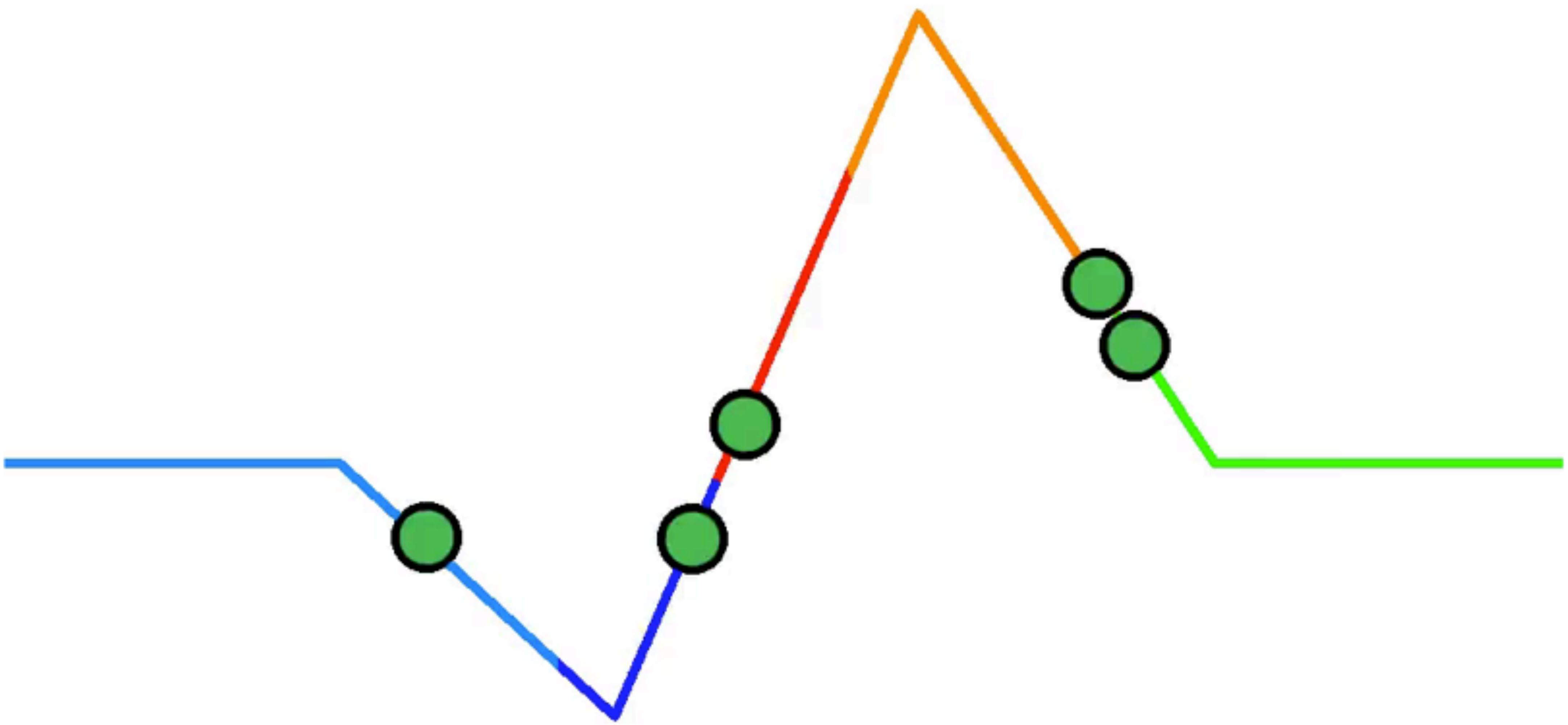
COMPUTE S_i

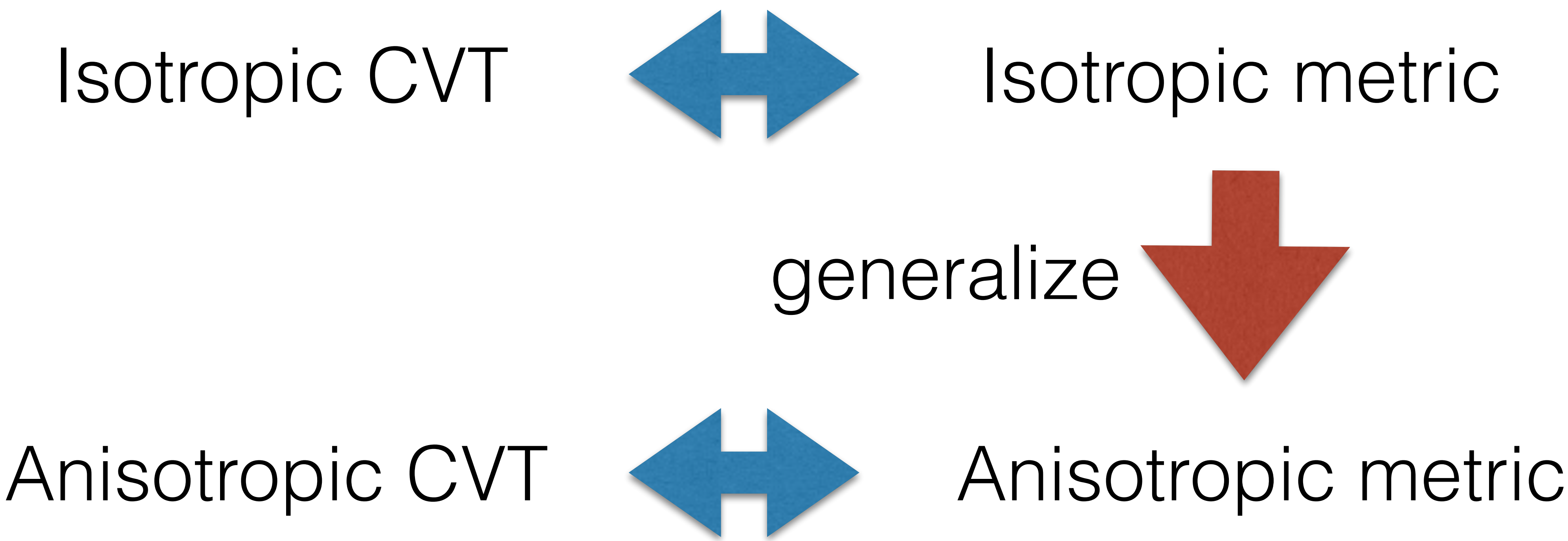
FIXED S_i

UPDATE $\mathbf{x}_i$

CVT

1

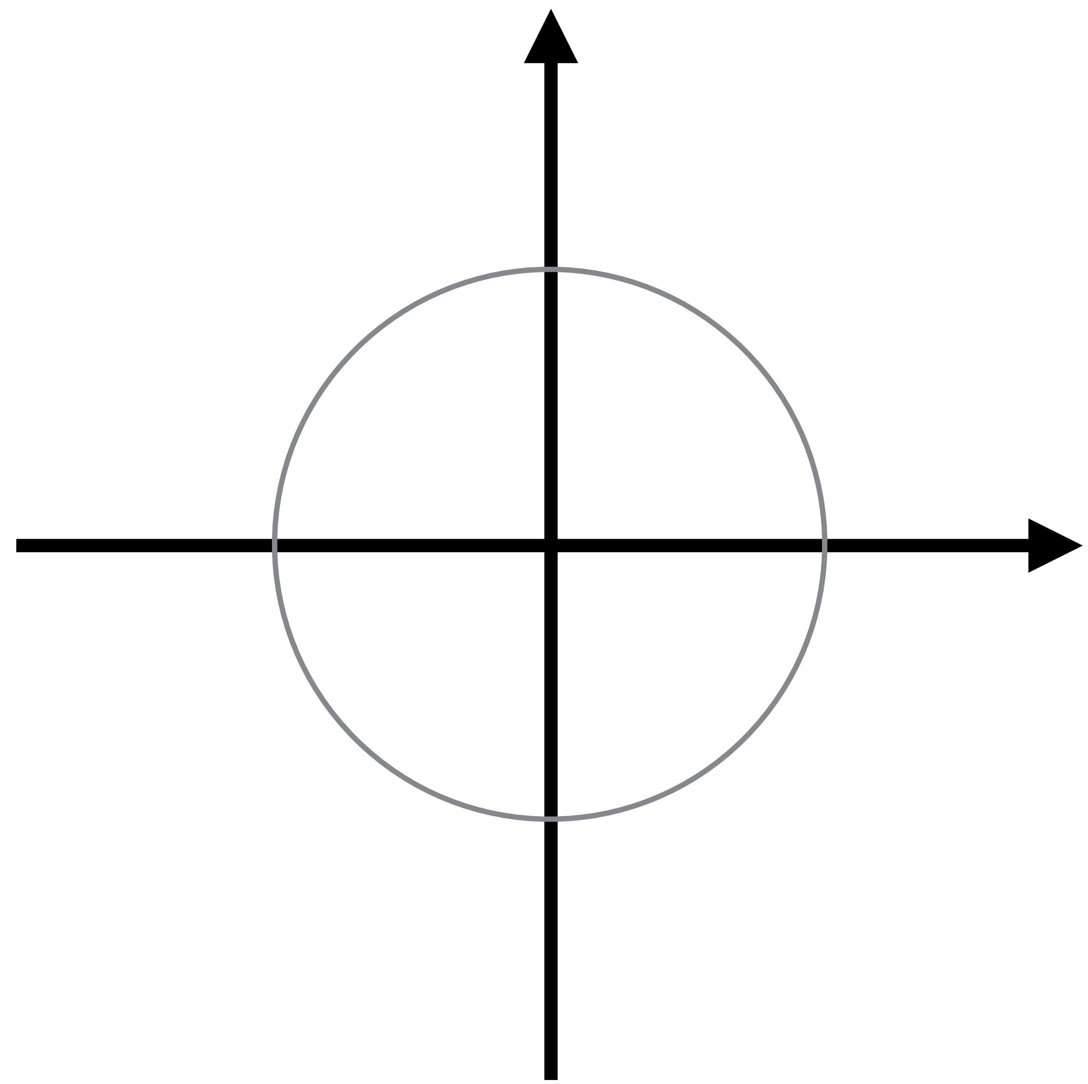



Isotropic CVT
Isotropic metric
generalize
Anisotropic CVT
Anisotropic metric

Isotropic metric

$$d(\mathbf{x}, \mathbf{p})^2 = (\mathbf{x} - \mathbf{p})^\mathsf{T}\mathbf{M}(\mathbf{x} - \mathbf{p})$$

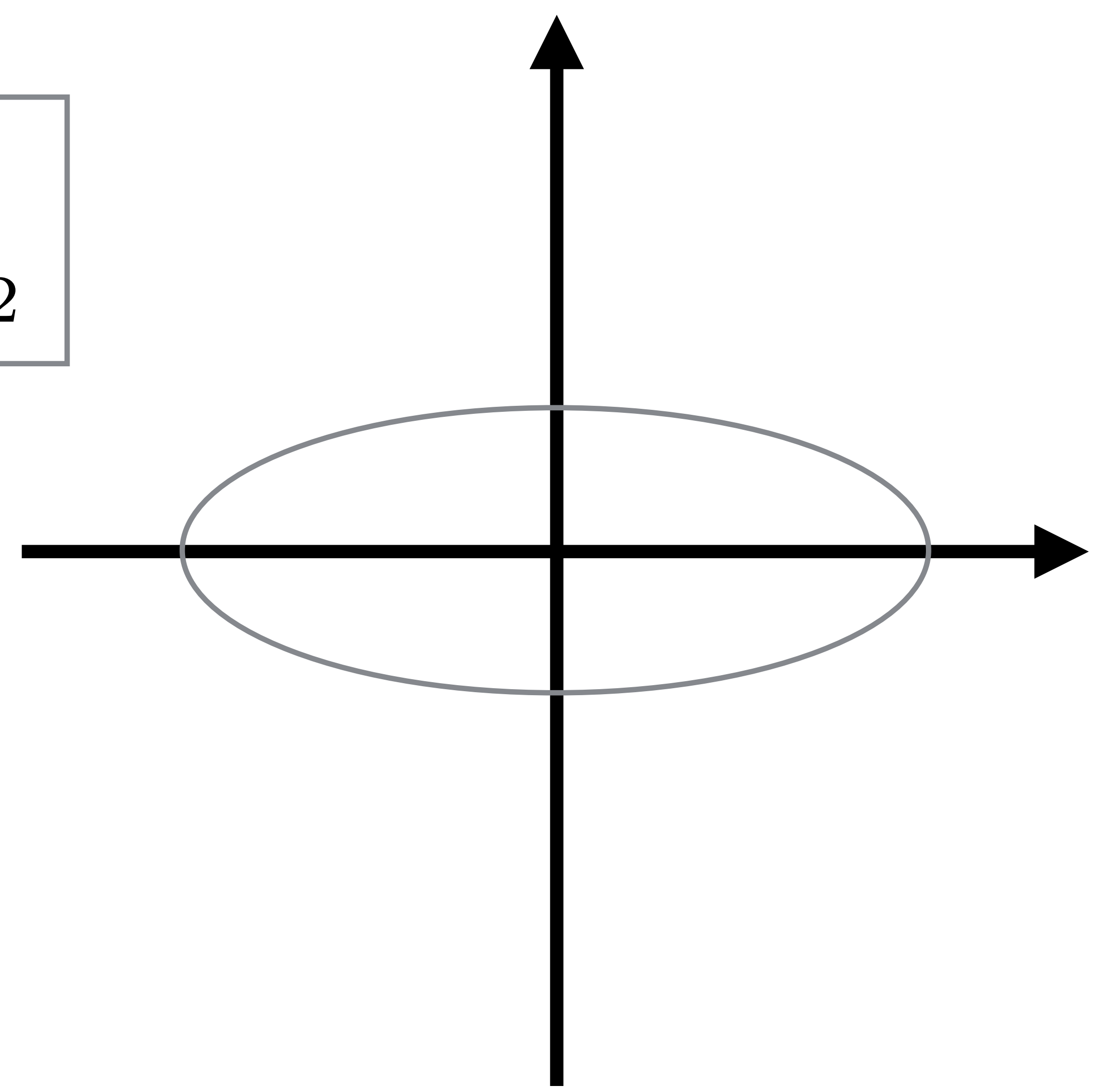
$$\mathbf{M} = \mathbf{I}$$



Anisotropic metric

$$d(\mathbf{x},\mathbf{p})^2 = (\mathbf{x}-\mathbf{p})^\mathsf{T}\mathbf{M}(\mathbf{x}-\mathbf{p})$$

$$\mathbf{M} = \begin{bmatrix} s_1 & \\ & s_2 \end{bmatrix}$$



Anisotropic CVT

Minimize:

$$F(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{M}_0, \ldots, \mathbf{M}_{k-1}) =$$

$$\sum_i \int_{S_i} (\mathbf{p} - \mathbf{x}_i)^\mathsf{T} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p}$$

Problem: trivial solutions

$$\mathbf{M}_i = \mathbf{0}$$

Solution: add constraint

$$|\mathbf{M}_i| = c_i > 0$$

Anisotropic CVT

Minimize:

$$L(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{M}_0, \ldots, \mathbf{M}_{k-1}) =$$

$$\sum_i \left(\underbrace{\int_{S_i} (\mathbf{p} - \mathbf{x}_i)^{\mathsf{T}} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p}} \;+\; \underbrace{\lambda |\mathbf{M}_i|} \right)$$

CVT energy

Lagrange multiplier constraining M

Anisotropic CVT

$$\sum_i \left(\int_{S_i} (\mathbf{p} - \mathbf{x}_i)^{\mathsf{T}} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \;+\; \lambda |\mathbf{M}_i| \right)$$

$$\frac{\partial L}{\partial \mathbf{x}_i} : \qquad \int_{S_i} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \qquad\qquad \mathbf{0}$$

$$\mathbf{x}_i^{new} = \frac{\int_{S_i} \mathbf{p}\, d\mathbf{p}}{\int_{S_i} 1\, d\mathbf{p}}$$

independent of M
(as for isotropic CVT)

Anisotropic CVT

$$\sum_i \left(\int_{S_i} (\mathbf{p} - \mathbf{x}_i)^{\mathsf{T}} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \;+\; \lambda |\mathbf{M}_i| \right)$$

$$\frac{\partial L}{\partial \mathbf{M}_i} : \quad \int_{S_i} (\mathbf{p} - \mathbf{x}_i)(\mathbf{p} - \mathbf{x}_i)^T d\mathbf{p} \qquad \lambda |\mathbf{M}_i| \; \left(\mathbf{M}_i^{-1}\right)^{\mathsf{T}}$$

$= \mathbf{C}_i$ *covariance matrix*

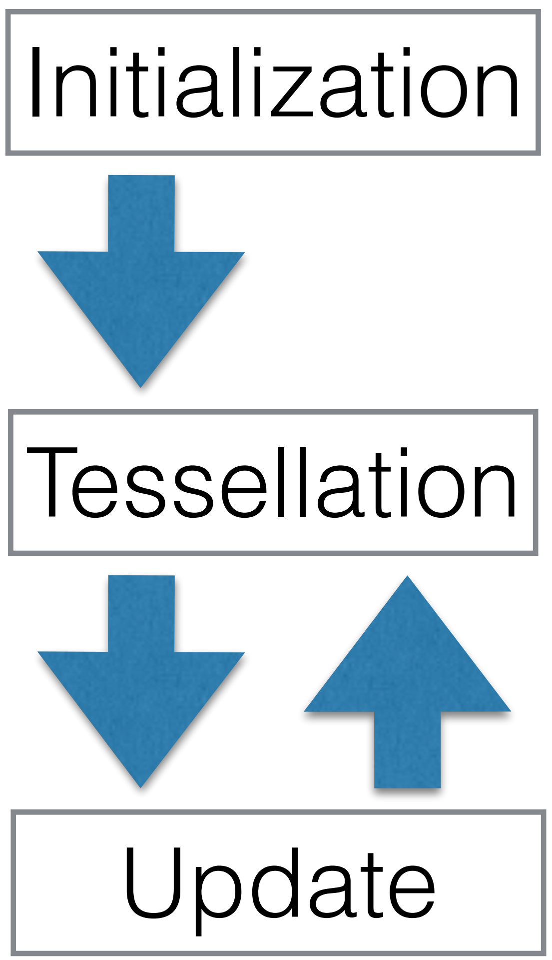
$$\mathbf{C}_i + \lambda |\mathbf{M}_i| \; \left(\mathbf{M}_i^{-1}\right)^{\mathsf{T}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{M}_i \propto \mathbf{C}_i^{-1}$$

Anisotropic CVT

$$\mathbf{C}_i = \mathbf{Q}_i \, \mathrm{diag}(\lambda_{i_2}, \lambda_{i_1}, \lambda_{i_0}) \mathbf{Q}_i^\mathsf{T}$$

$$\mathbf{M}_i = \lambda_{i_0} \mathbf{C}^{-1} = \mathbf{Q}_i \, \mathrm{diag} \left(\frac{\lambda_{i_0}}{\lambda_{i_2}}, \frac{\lambda_{i_0}}{\lambda_{i_1}}, 1 \right) \mathbf{Q}_i^\mathsf{T}$$

Algorithm (Lloyd)



$\mathbf{x}_i = rand(S)$

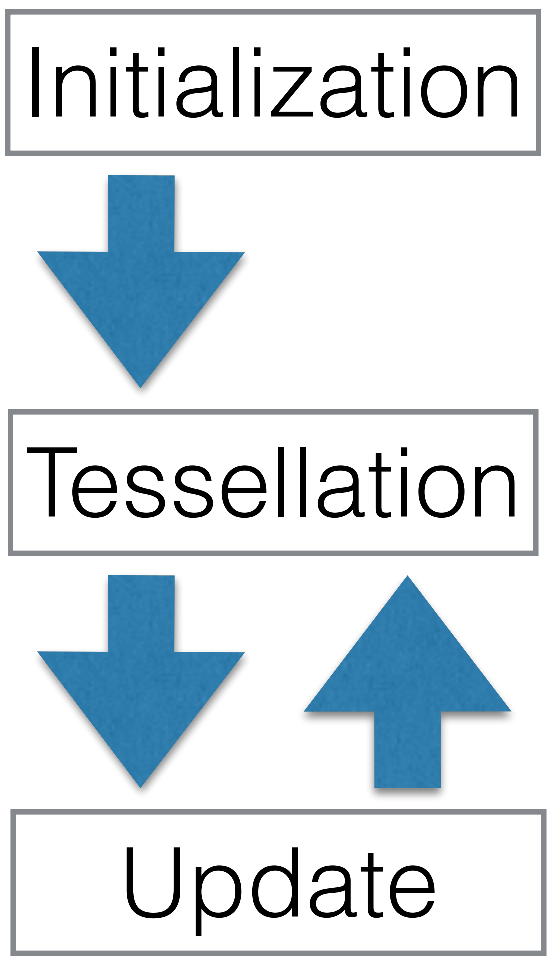
FIXED $\mathbf{x}_i$

COMPUTE S_i

FIXED S_i

UPDATE $\mathbf{x}_i$

Algorithm



$\mathbf{x}_i = rand(S)$

$\mathbf{M}_i = \mathbf{I}$

FIXED $\mathbf{x}_i, \mathbf{M}_i$

COMPUTE S_i

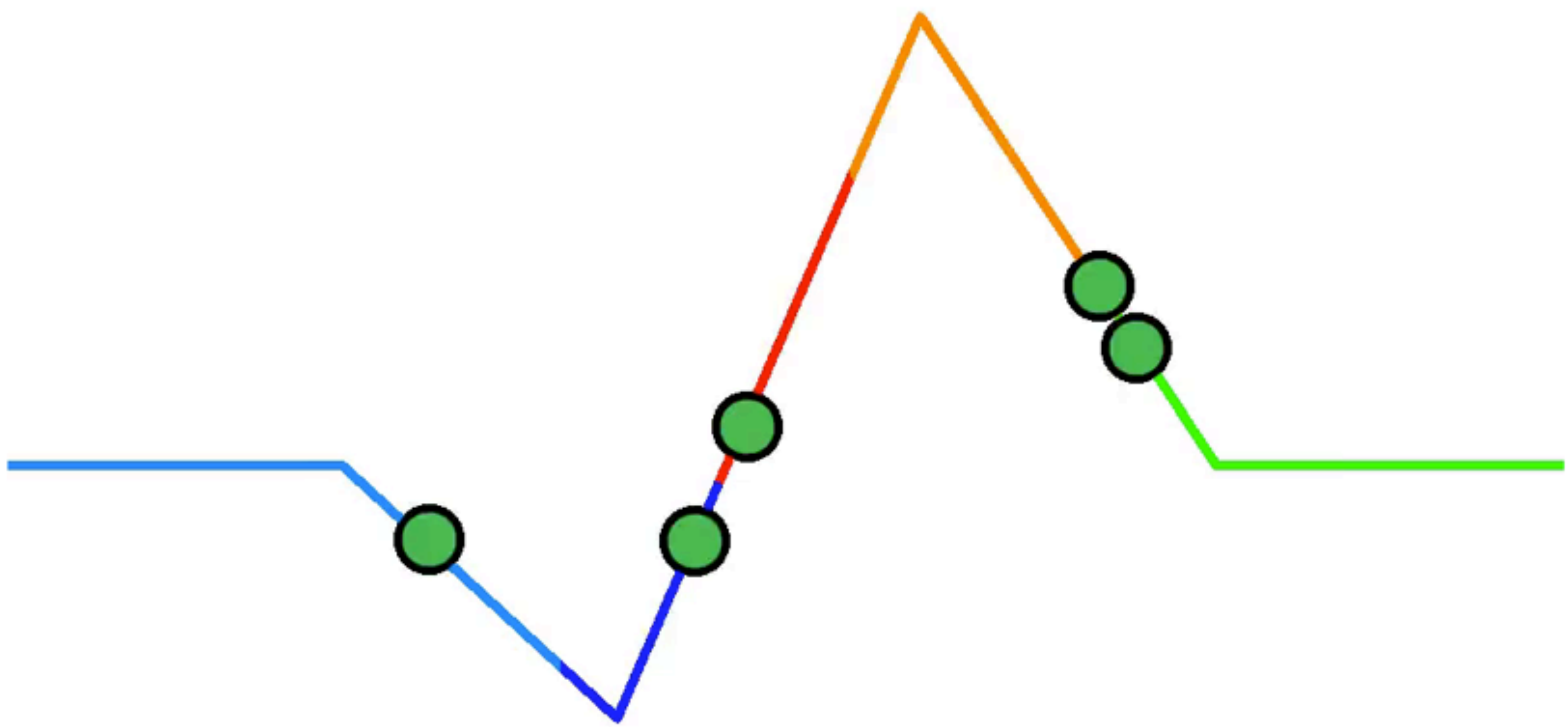
FIXED S_i

UPDATE $\mathbf{x}_i$

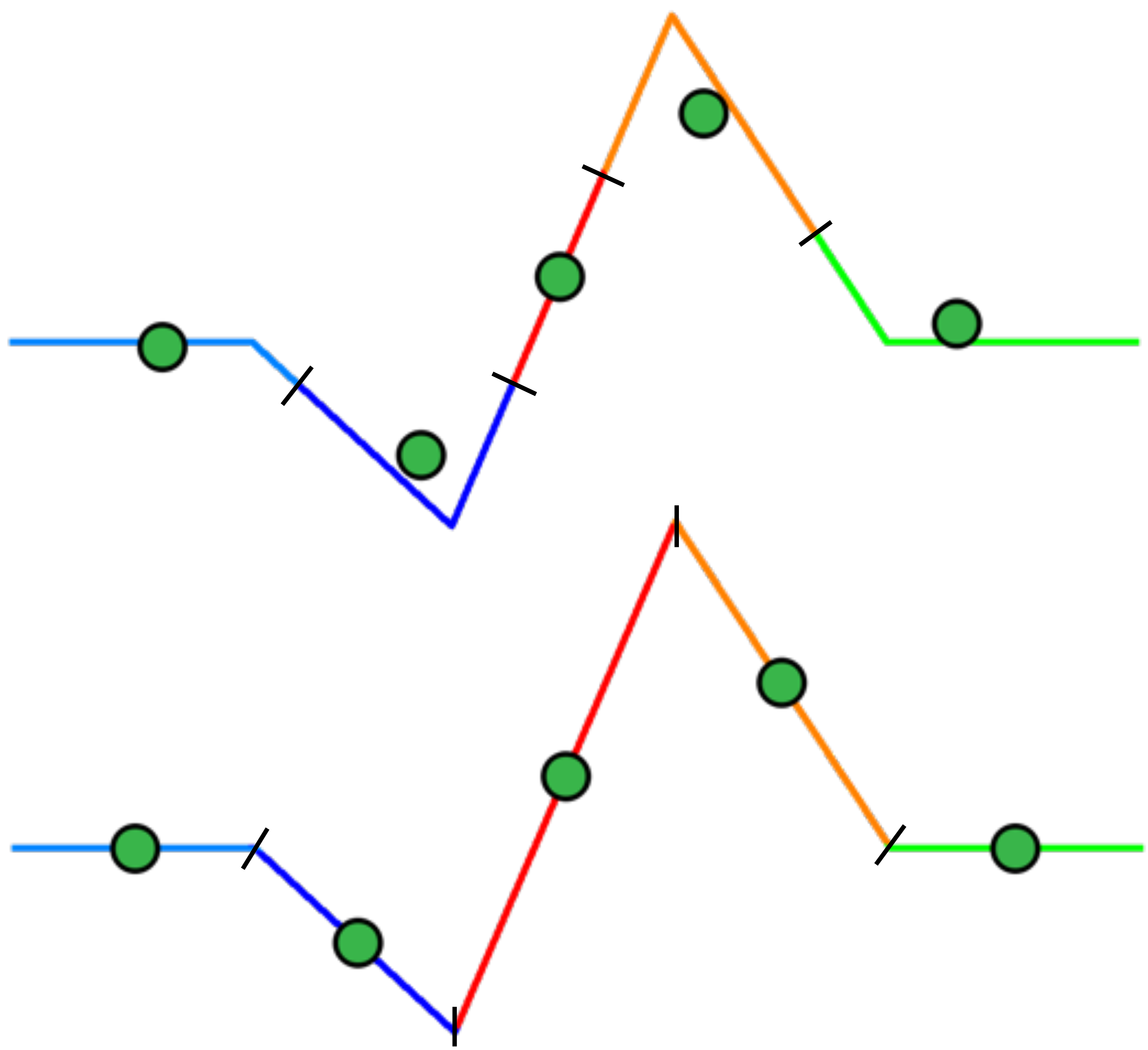
UPDATE $\mathbf{M}_i$

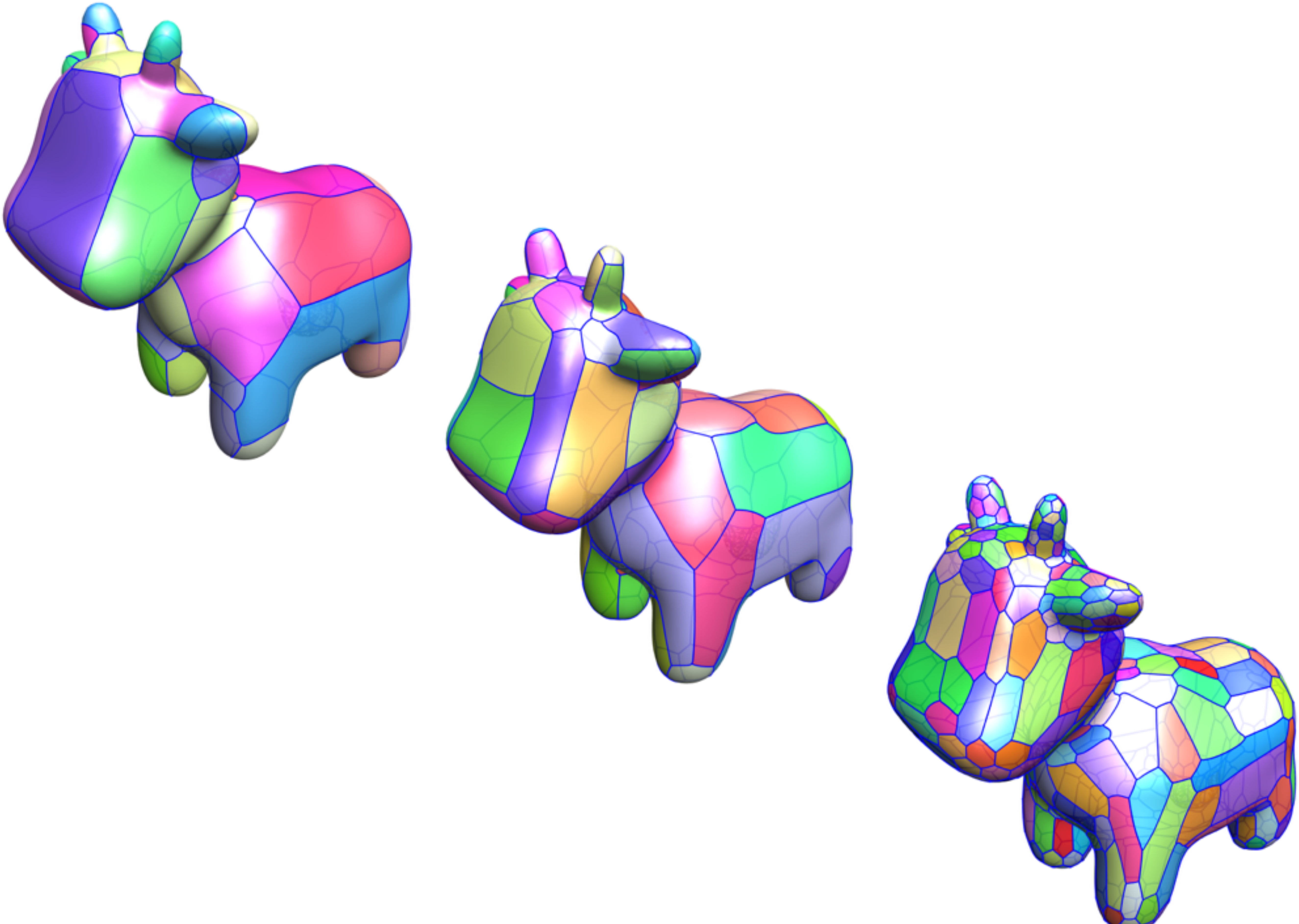
Anisotropic CVT

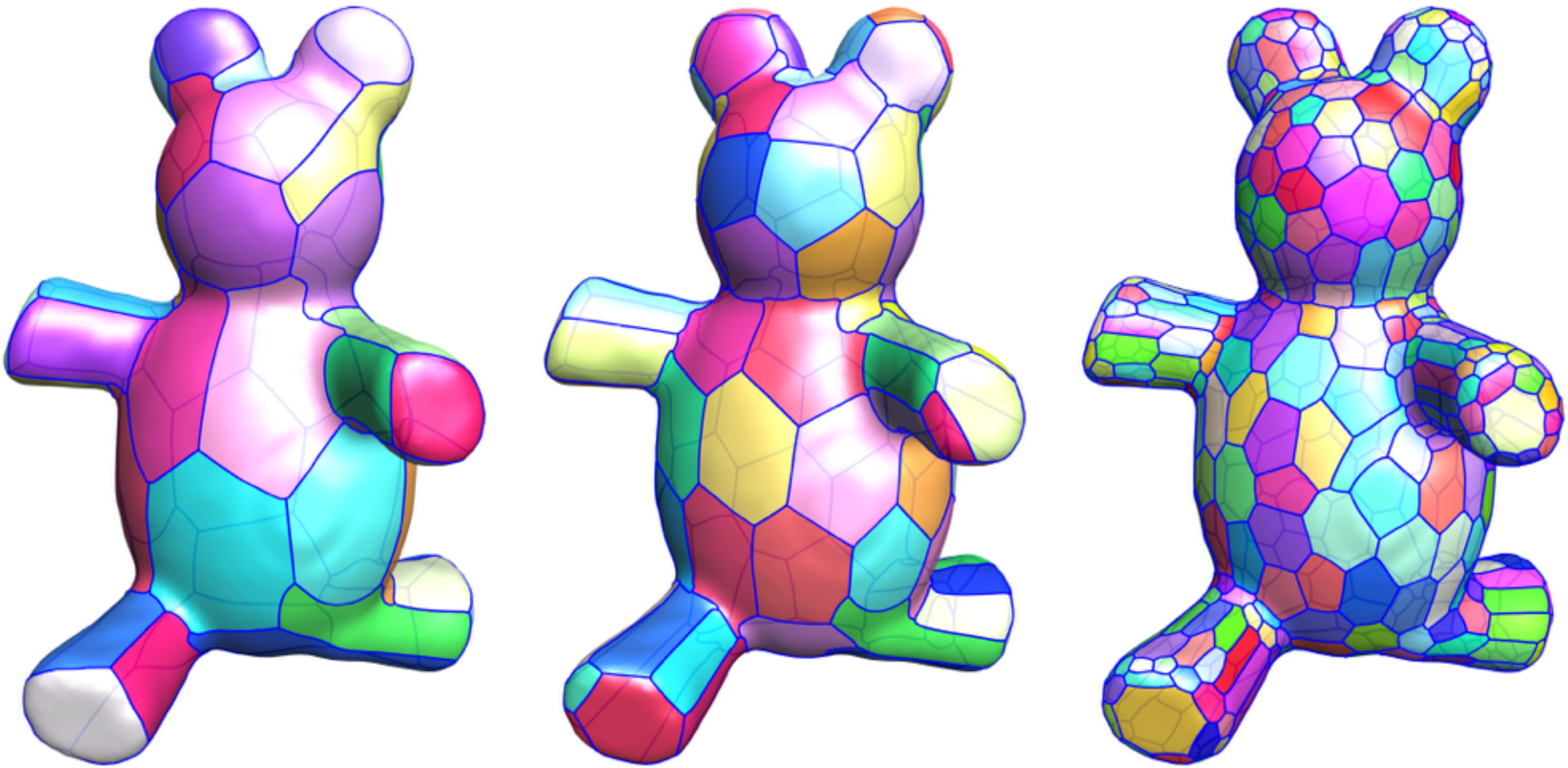
1

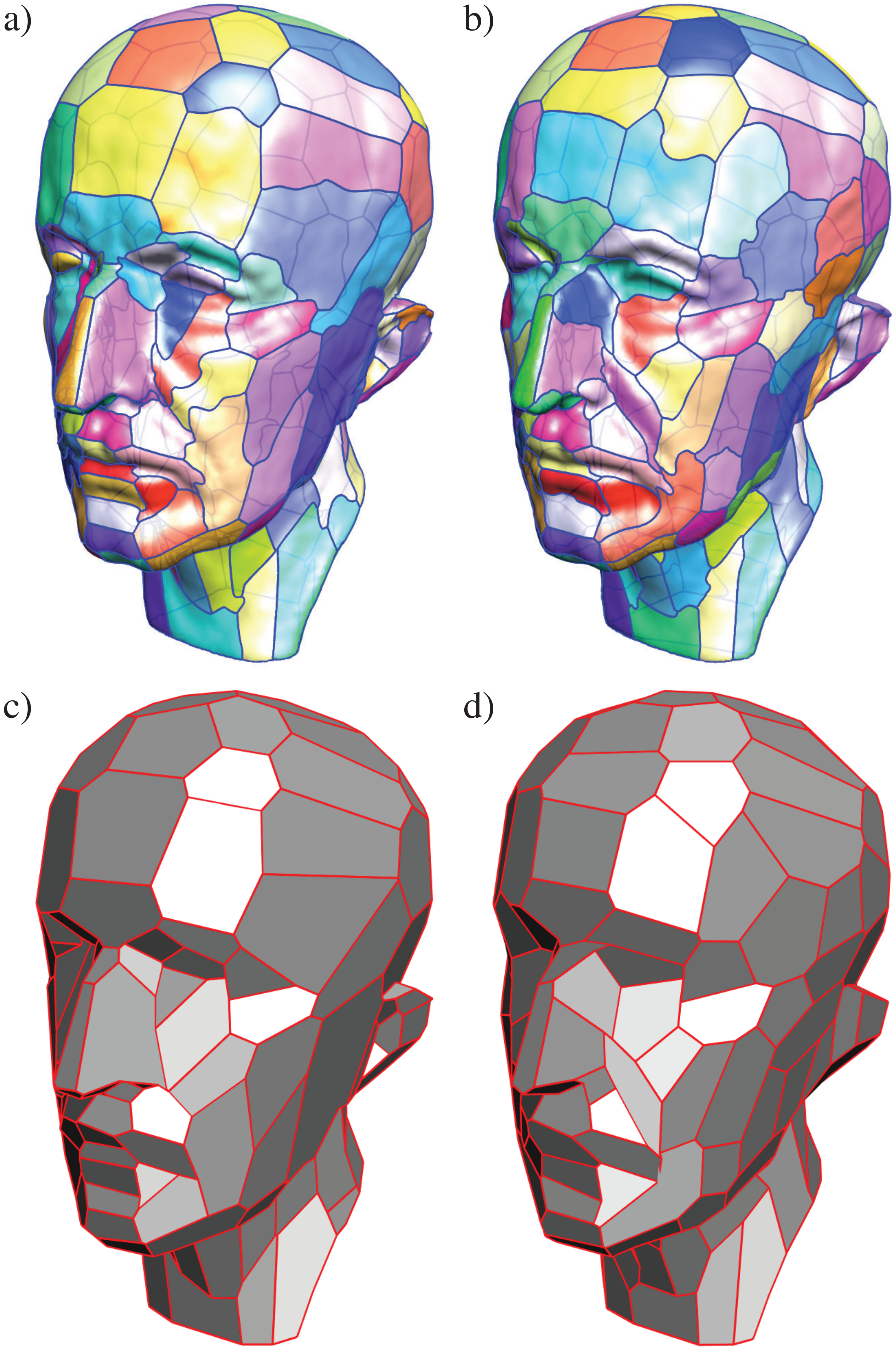


Isotropic vs. Anisotropic



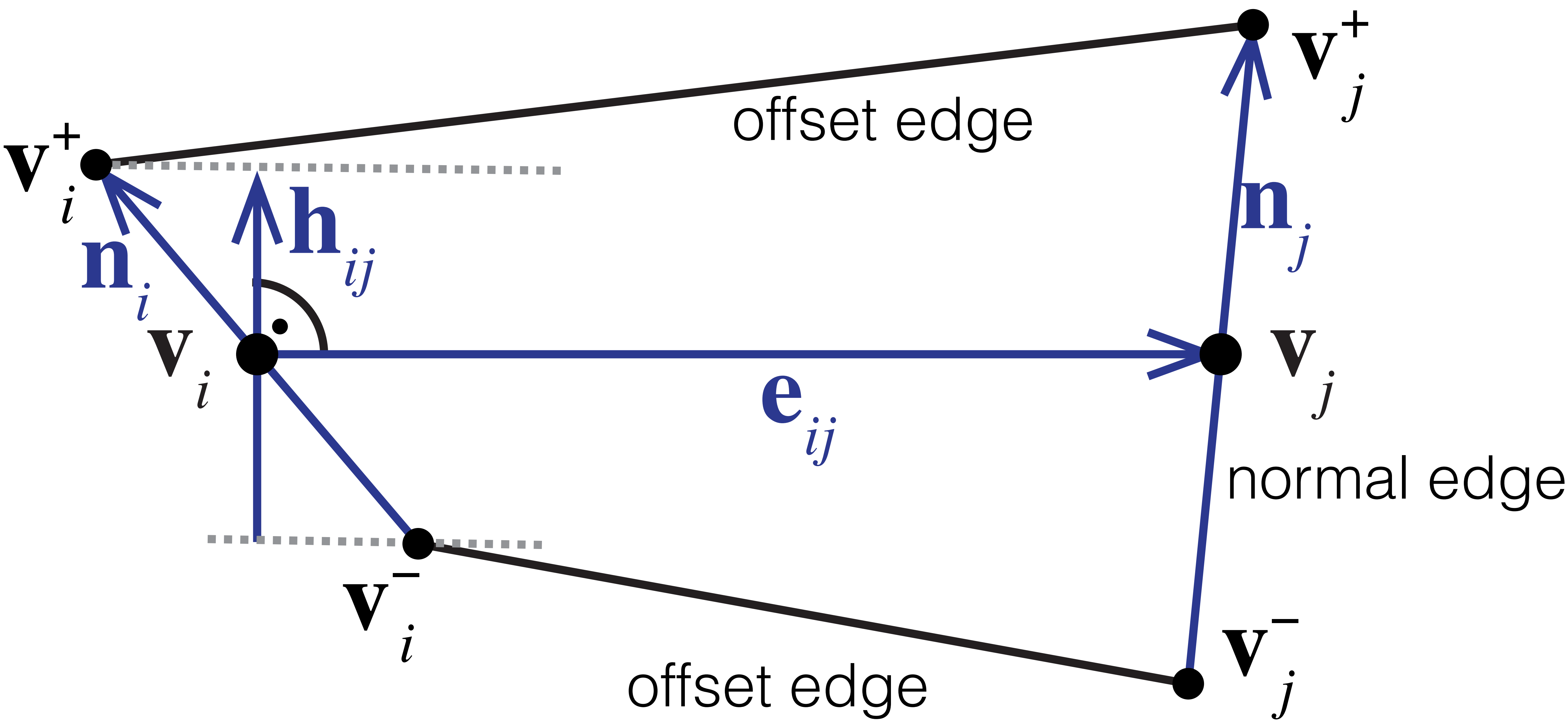





a)
b)
c)
d)

Shaping Up the Beams

Beam quad



$$\mathbf{e}_{ij}^{\pm} = \mathbf{v}_j^{\pm} - \mathbf{v}_i^{\pm} = \mathbf{e}_{ij} \pm (\mathbf{n}_j - \mathbf{n}_i)$$

Algorithm (ARAP, ShapeUP)

Initialization

$$\mathbf{v}_i^{\pm} = \mathbf{v}_i \pm \frac{\mathbf{n}_i^*}{\|\mathbf{n}_i^*\|}.$$

Local (non-linear projection)

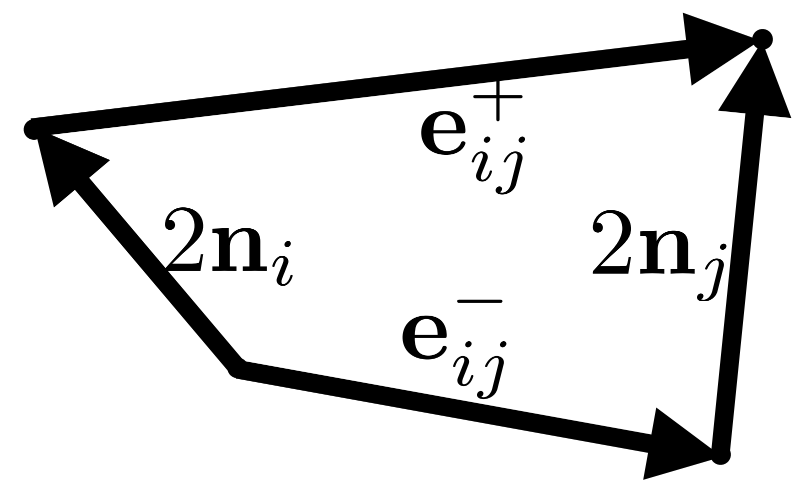
FIXED $\mathbf{v}_i^{\pm}$

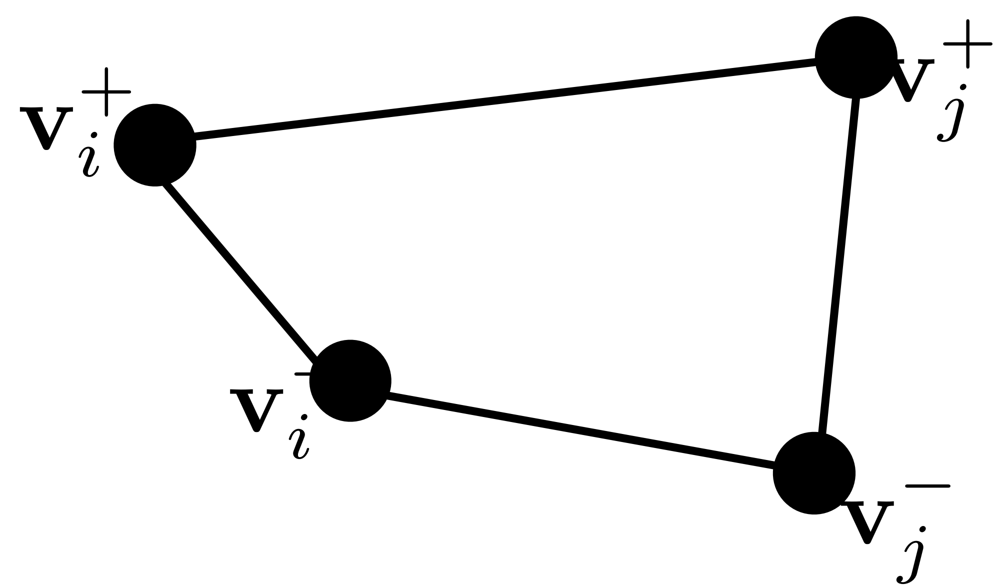
PROJECT $\mathbf{e}_{ij}^{\pm},\ \mathbf{n}_i$

Global (linear least squares)

FROM $\mathbf{e}_{ij}^{\pm},\ \mathbf{n}_i$

COMPUTE $\mathbf{v}_i^{\pm}$

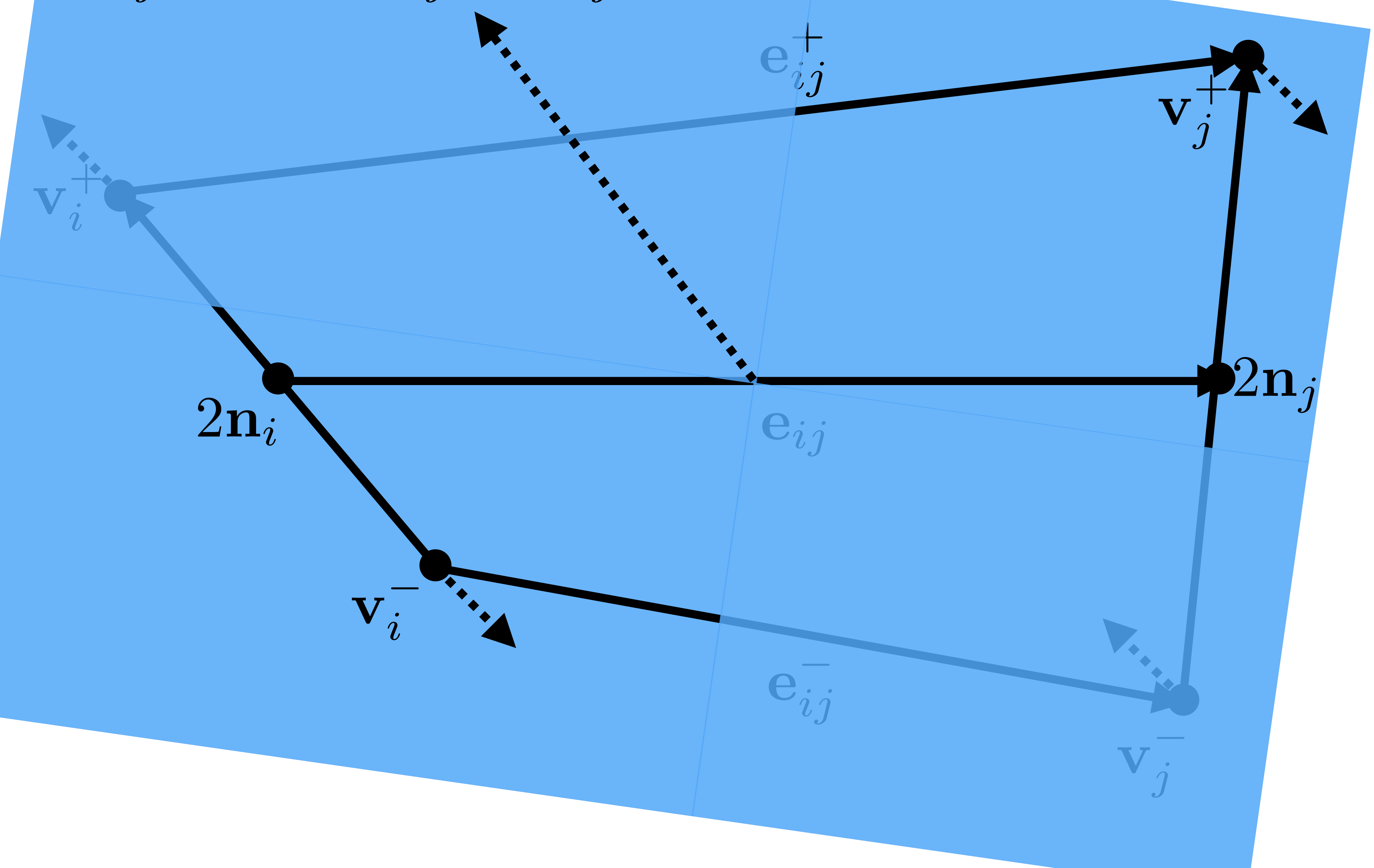


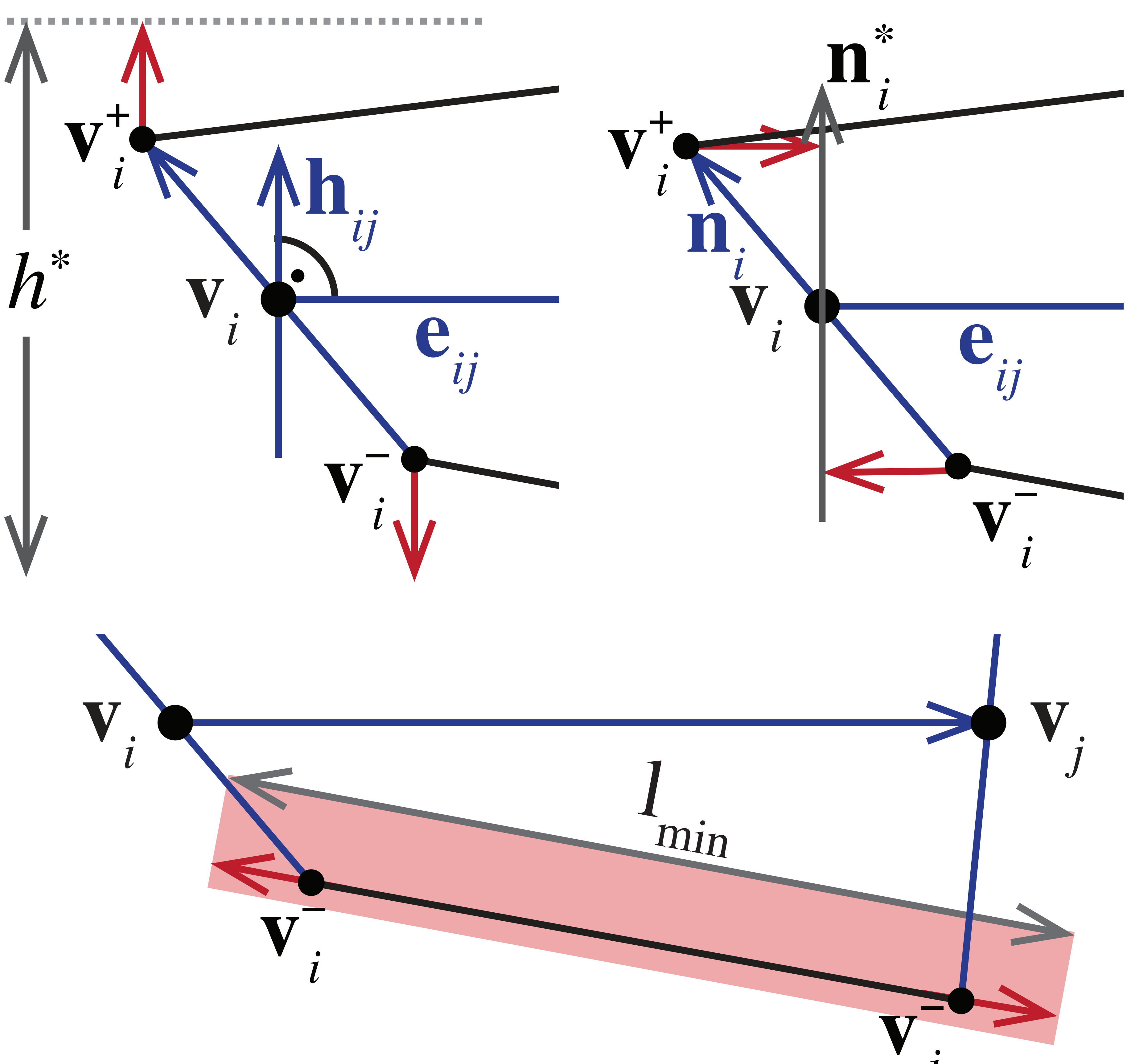


Planarity

$$\mathbf{v}^{\pm}_{i,j} - \frac{\mathbf{n}_{ij}^{\top}\mathbf{v}^{\pm}_{i,j}}{\mathbf{n}_{ij}^{\top}\mathbf{n}_{ij}}\mathbf{n}_{ij}$$

$$\mathbf{n}_{ij} = (\mathbf{n}_i + \mathbf{n}_j) \times \mathbf{e}_{ij}.$$





h*
v+i
hij
vi
eij
v−i
n*i
v+i
ni
vi
eij
v−i
vi
vj
lmin
v−i
v−j

Algorithm

Initialization

$$\mathbf{v}_i^{\pm} = \mathbf{v}_i \pm \frac{\mathbf{n}_i^*}{\|\mathbf{n}_i^*\|}.$$

Local (non-linear projection)

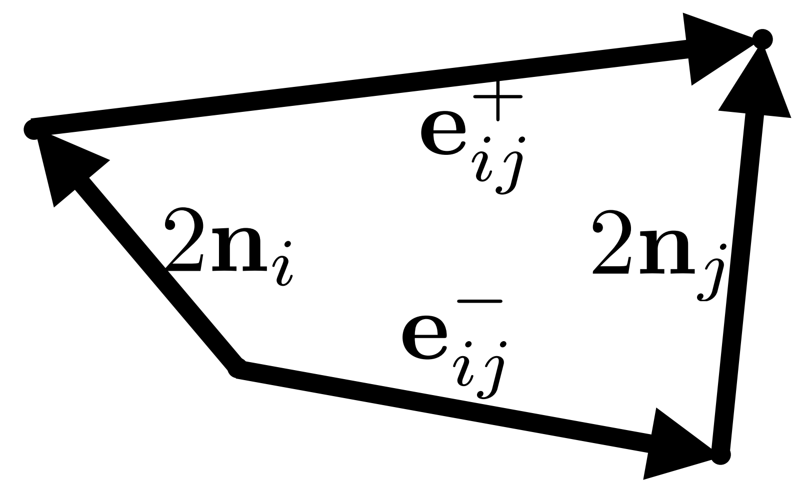
FIXED $\mathbf{v}_i^{\pm}$

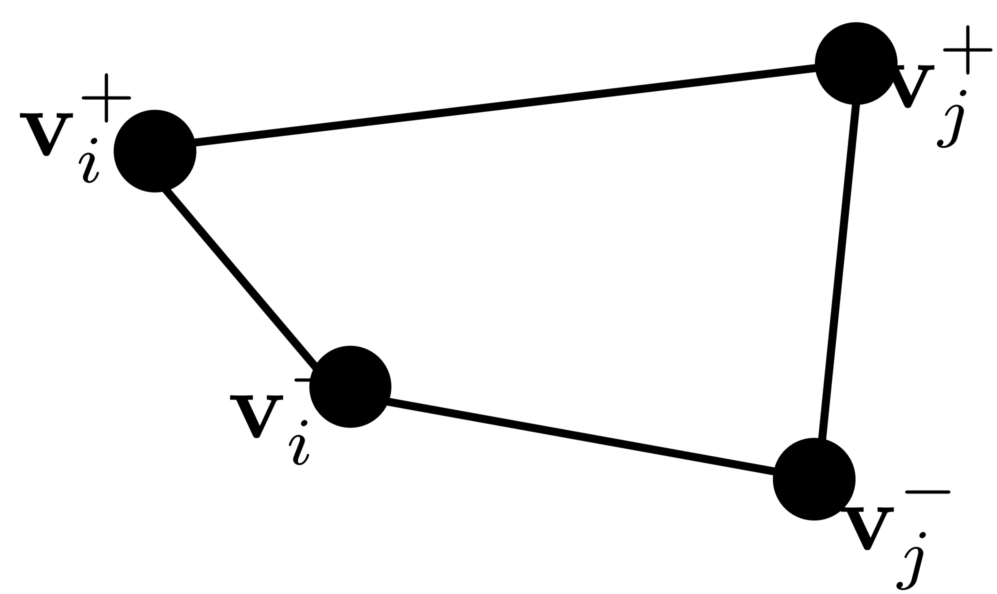
PROJECT $\mathbf{e}_{ij}^{\pm}, \ \mathbf{n}_i$

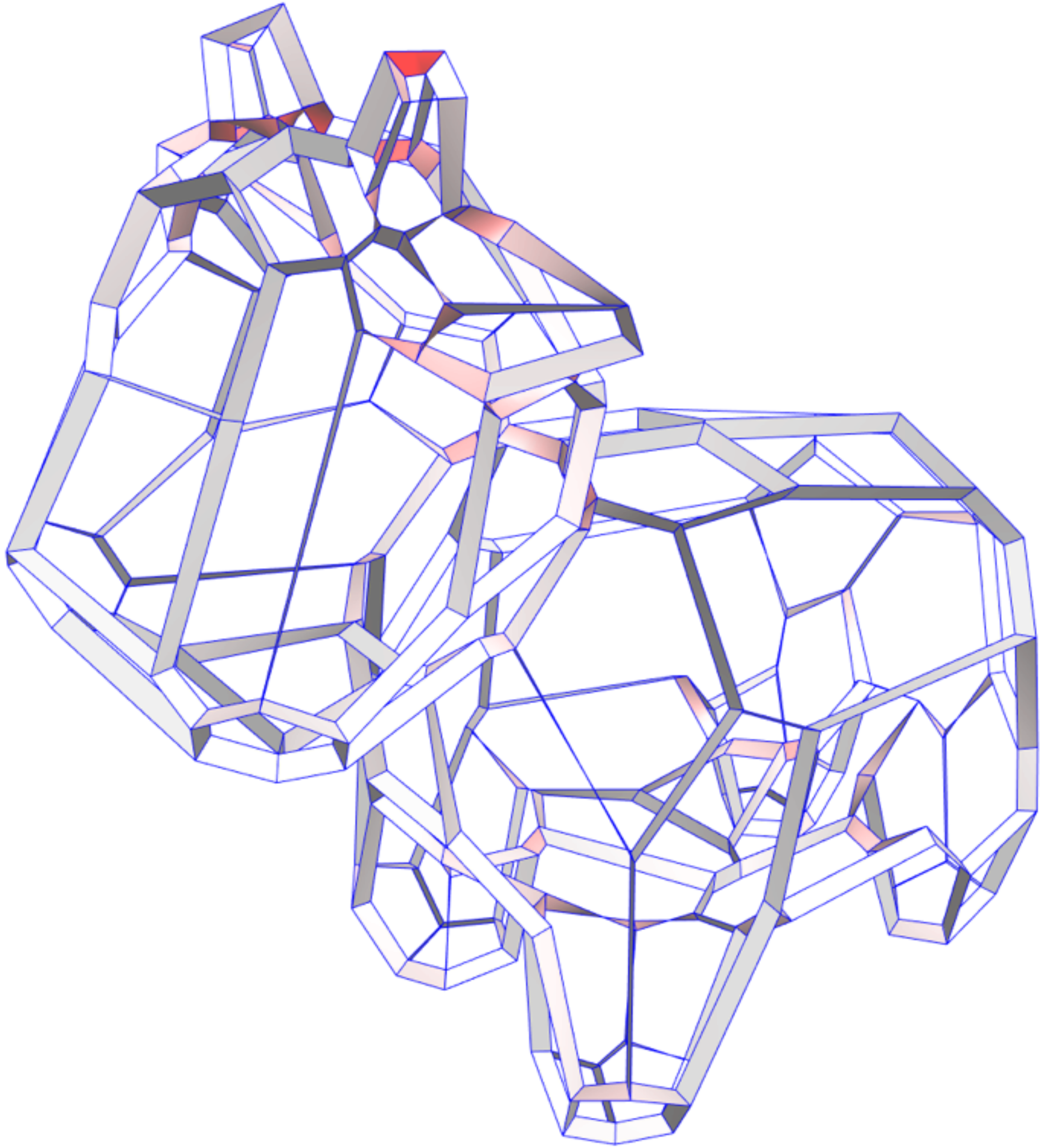
Global (linear least squares)

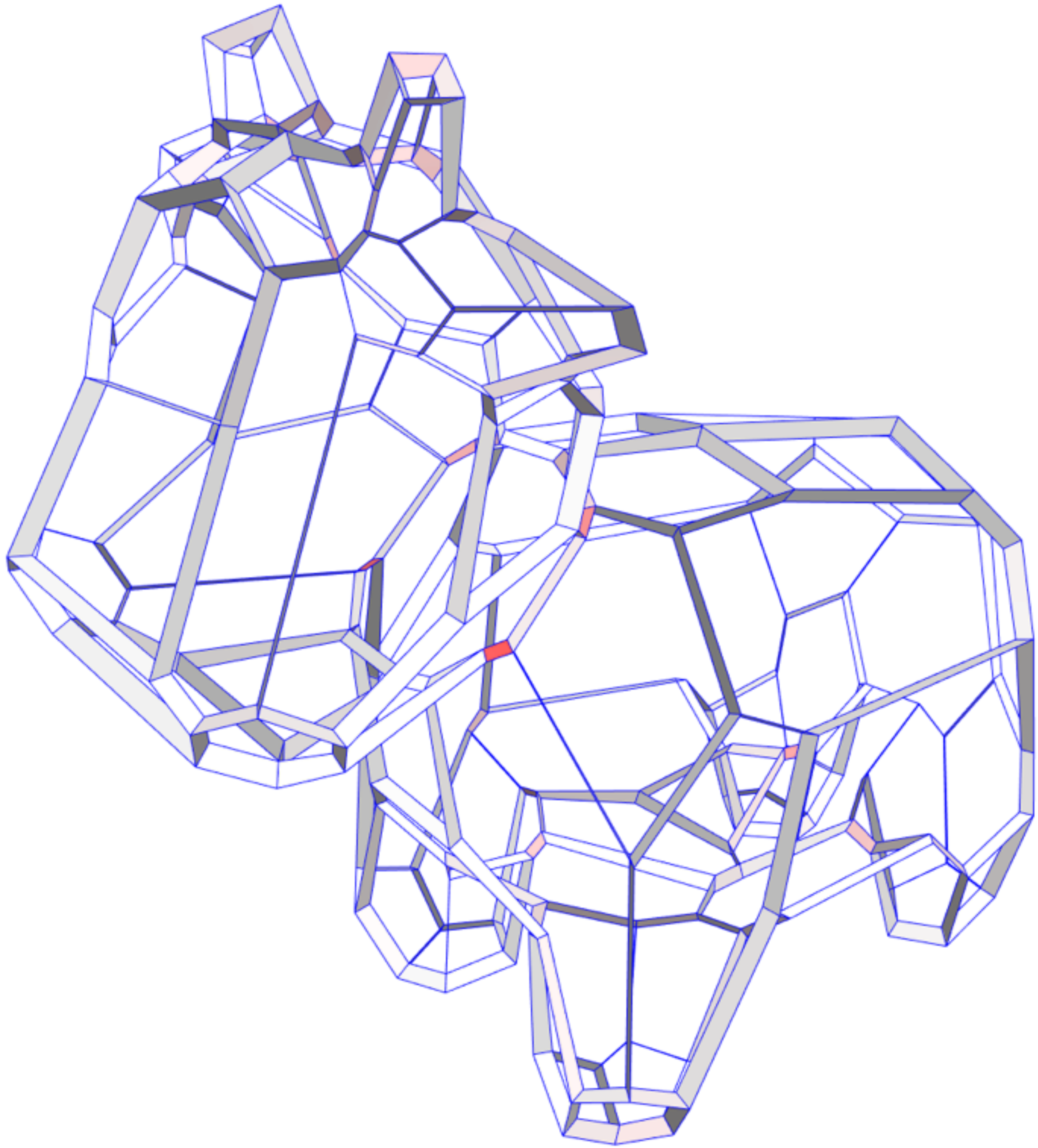
FROM $\mathbf{e}_{ij}^{\pm}, \ \mathbf{n}_i$

COMPUTE $\mathbf{v}_i^{\pm}$

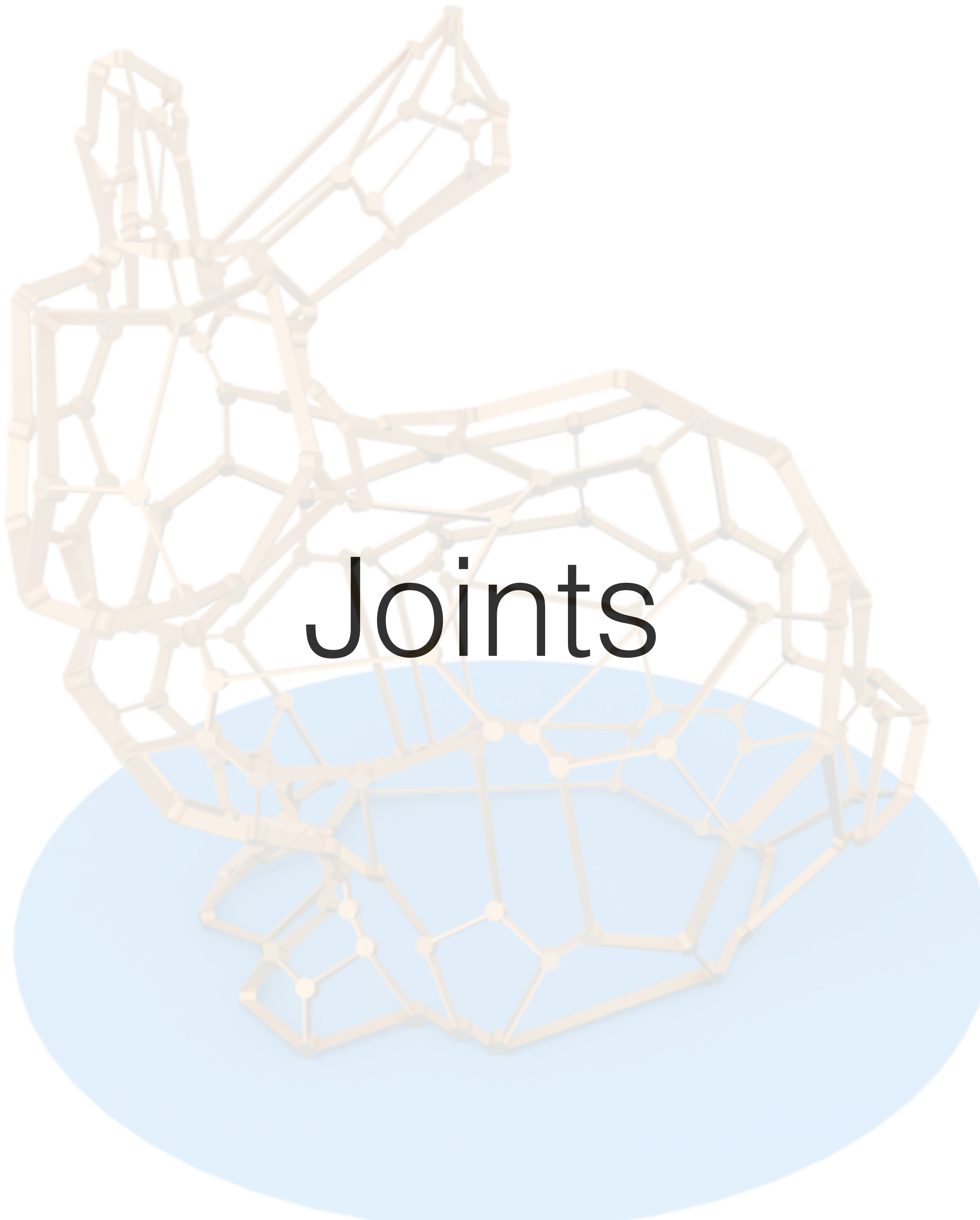






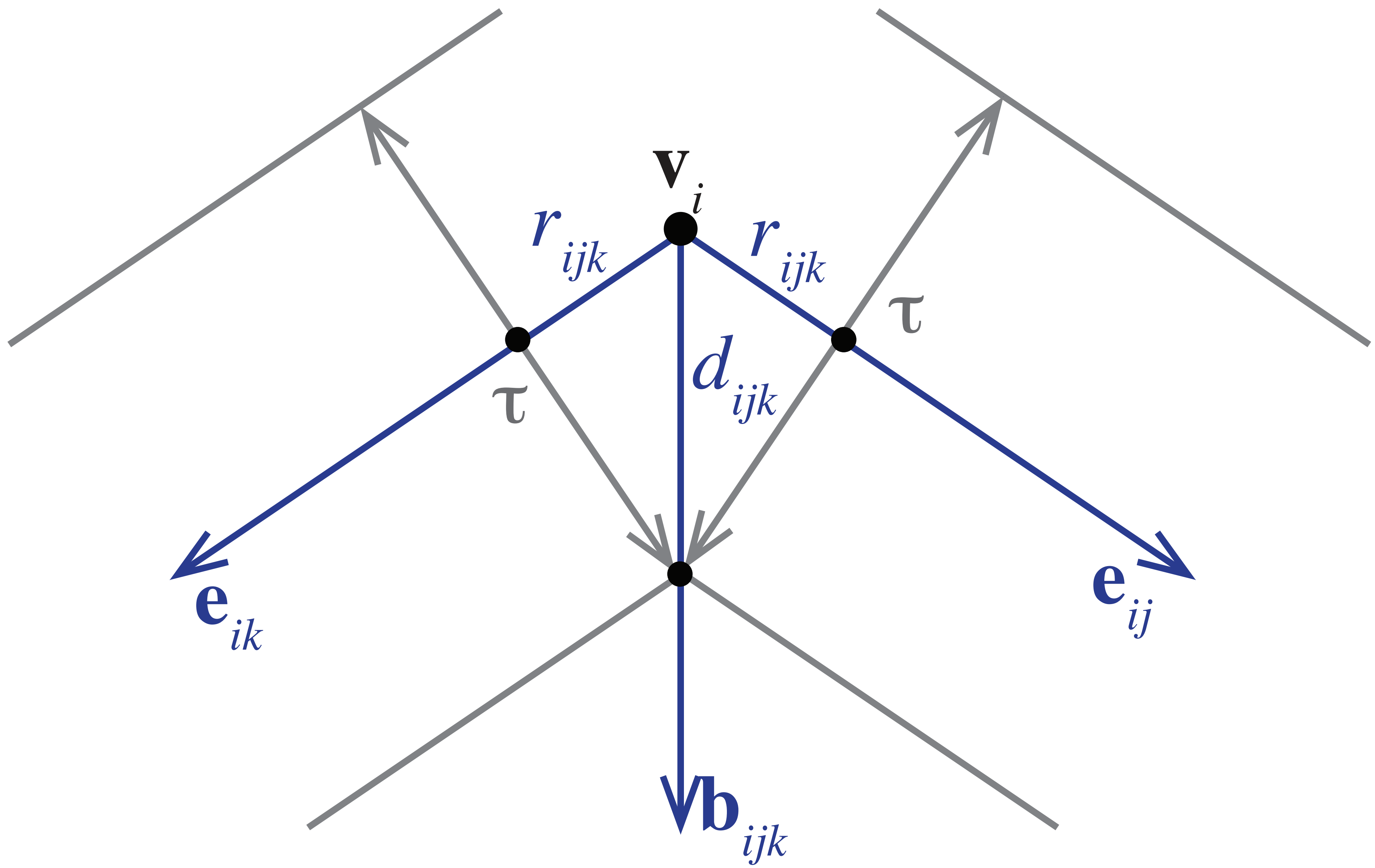


Joints

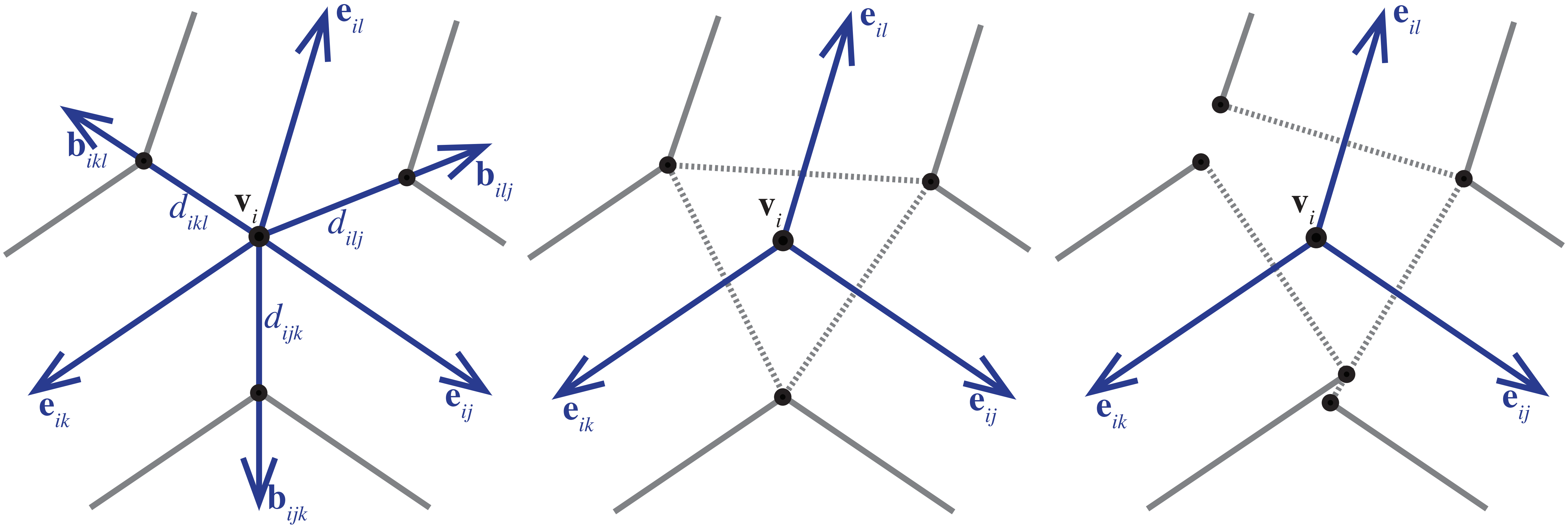


2 Beams

$$r_{ijk}\hat{\mathbf{e}}_{ij} + \tau\hat{\mathbf{e}}_{ij}^{\perp} = d_{ijk}\hat{\mathbf{b}}_{ijk}$$

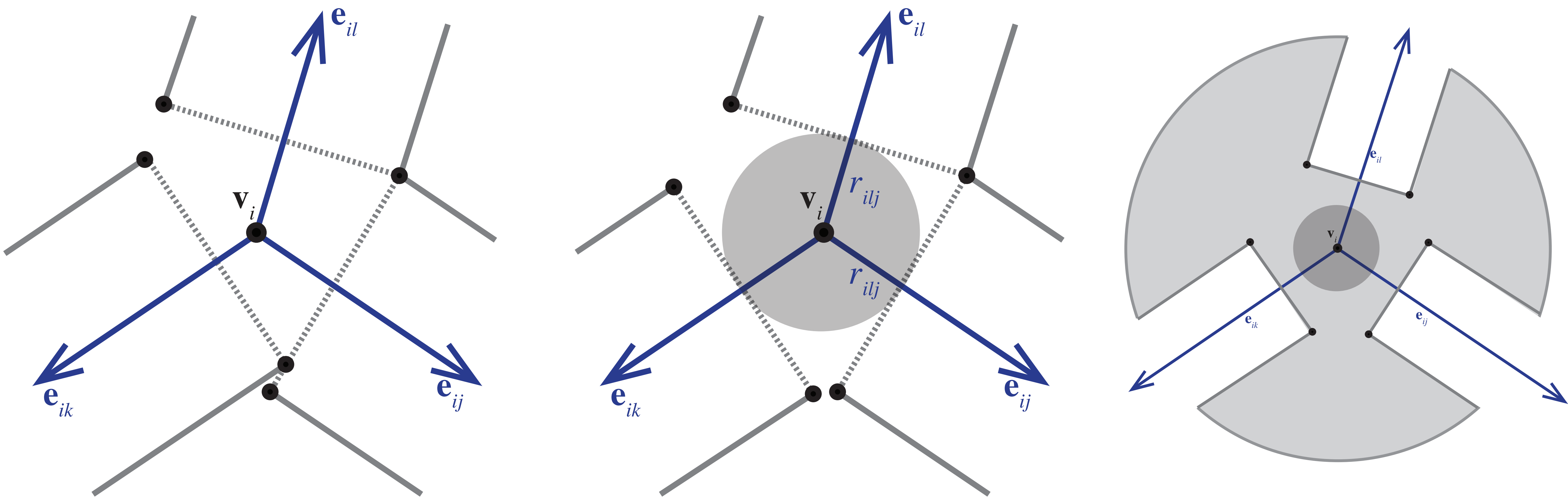


3 Beams

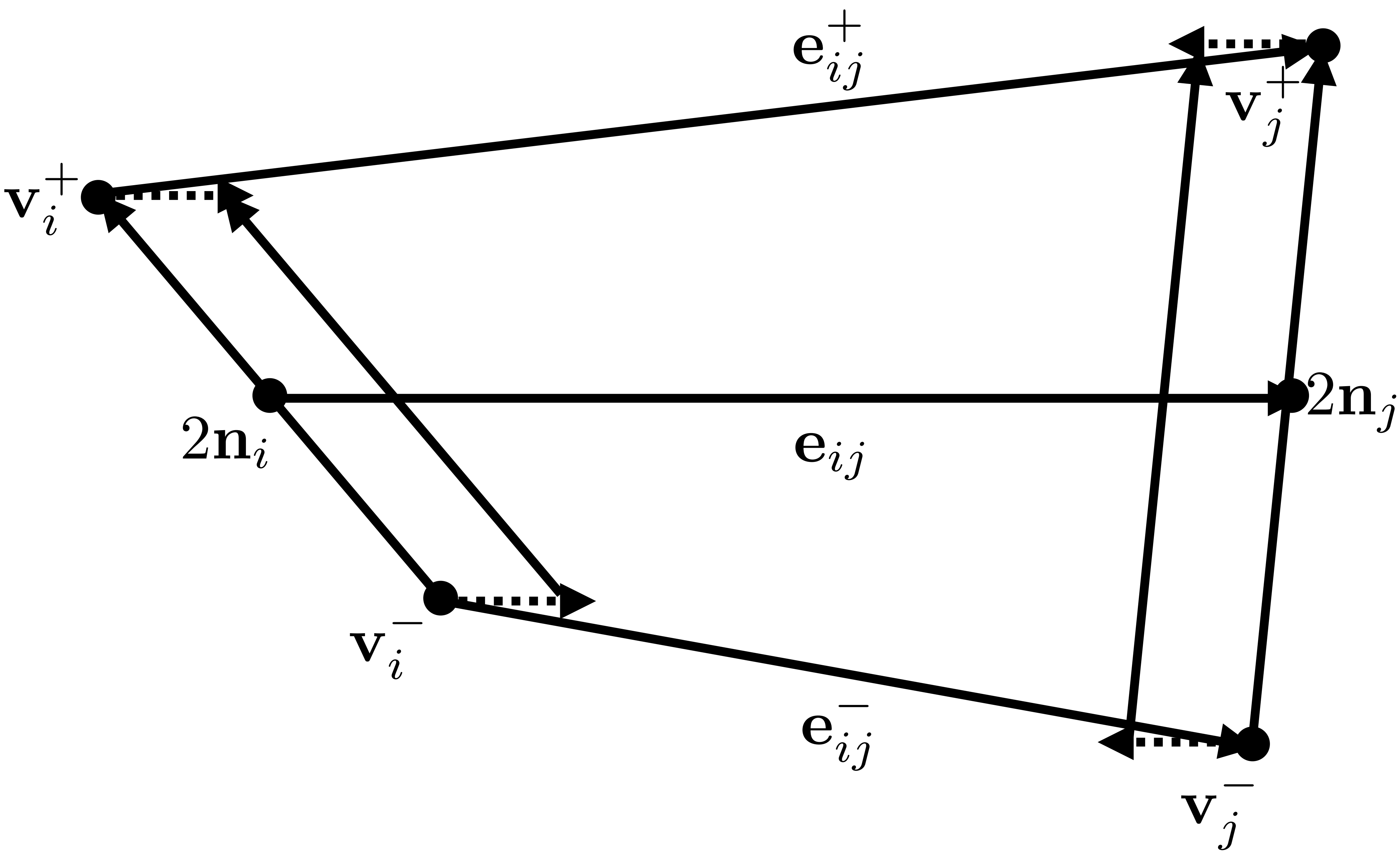


$\mathbf{e}_{il}$
$\mathbf{b}_{ikl}$
$\mathbf{b}_{ilj}$
$\mathbf{v}_i$
d_{ikl}
d_{ilj}
d_{ijk}
$\mathbf{e}_{ik}$
$\mathbf{e}_{ij}$
$\mathbf{b}_{ijk}$
$\mathbf{e}_{il}$
$\mathbf{v}_i$
$\mathbf{e}_{ik}$
$\mathbf{e}_{ij}$
$\mathbf{e}_{il}$
$\mathbf{v}_i$
$\mathbf{e}_{ik}$
$\mathbf{e}_{ij}$

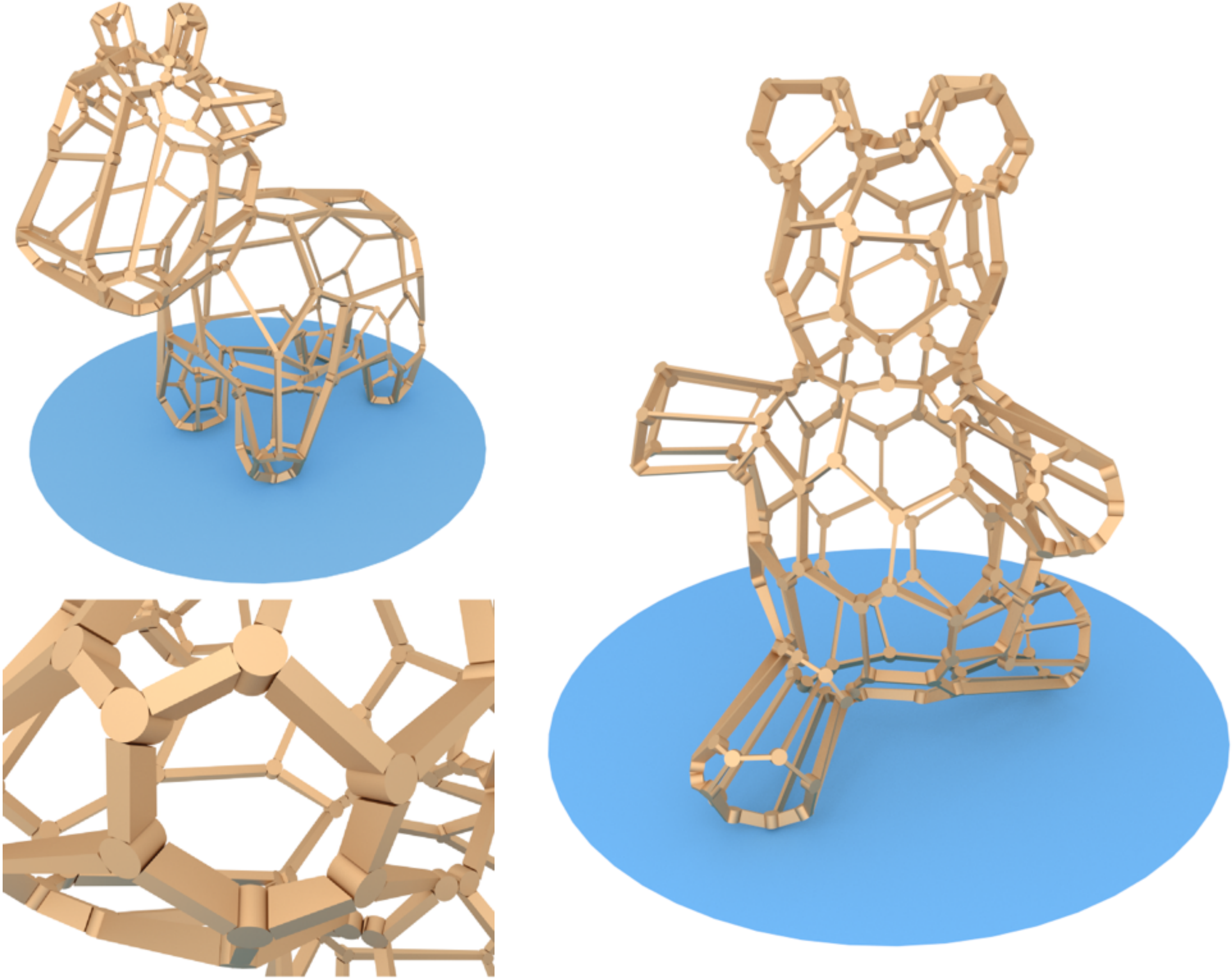
3 Beams



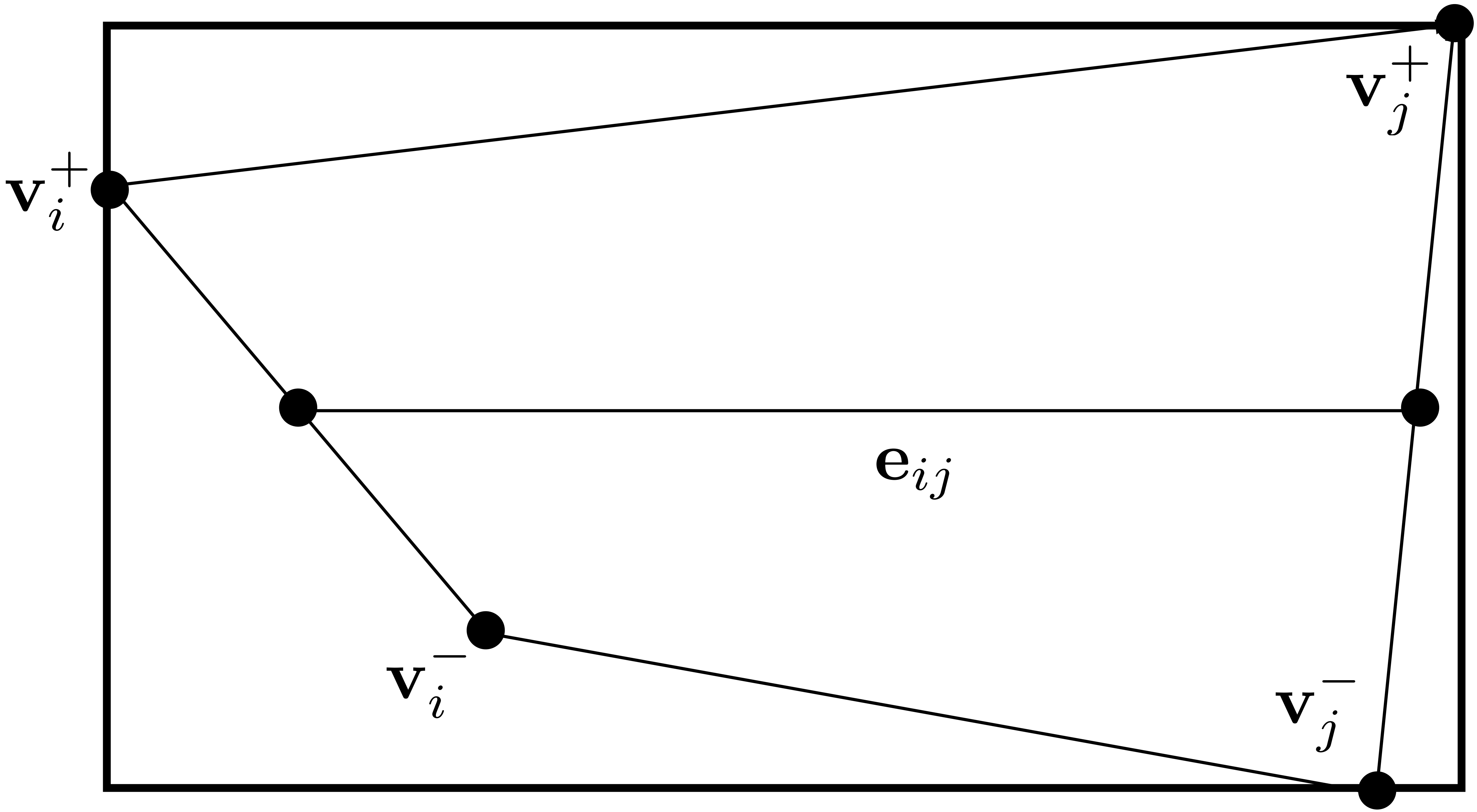
$$r_i = \max\{r_{ijk}, r_{ikl}, r_{ilj}\}$$

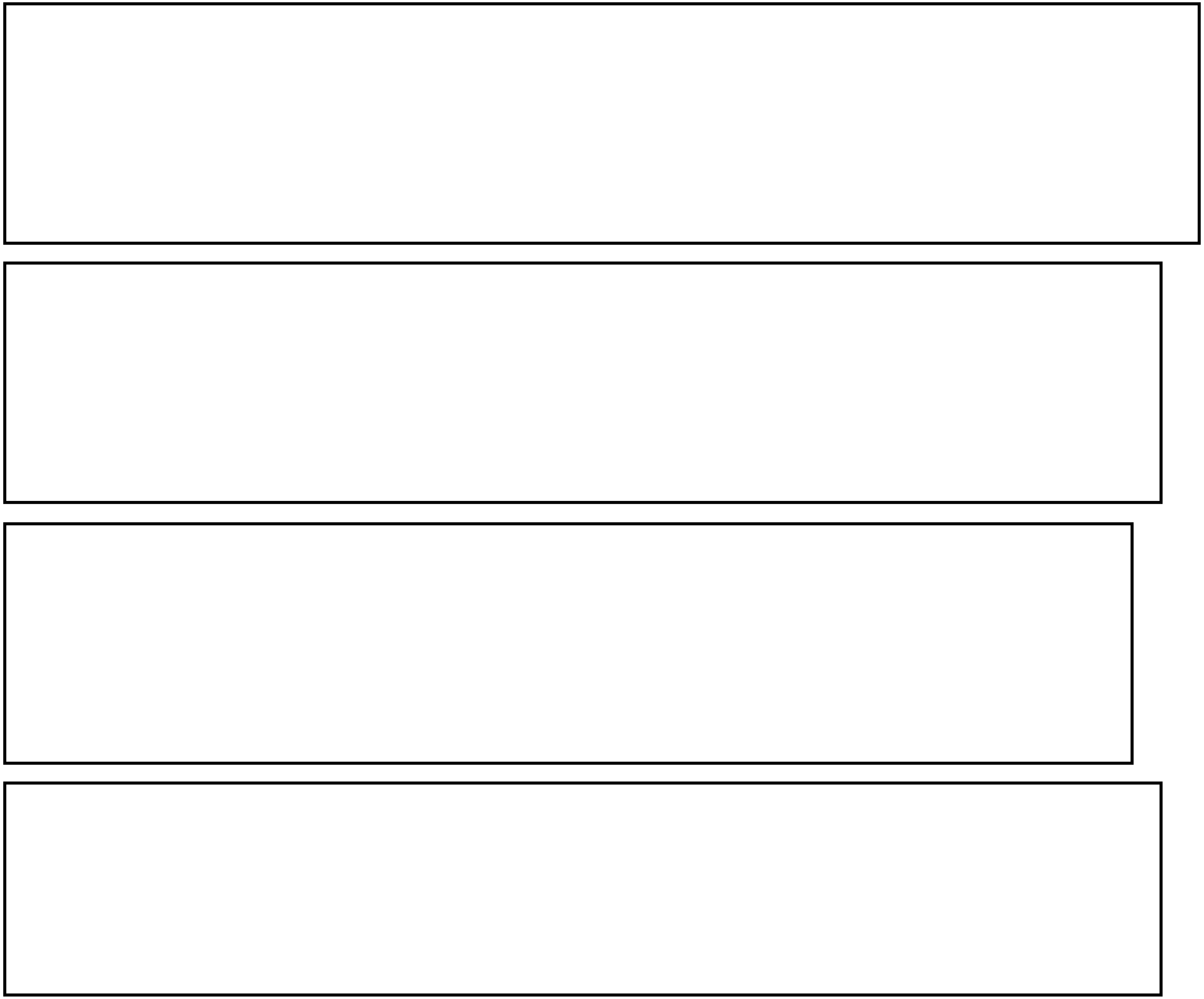


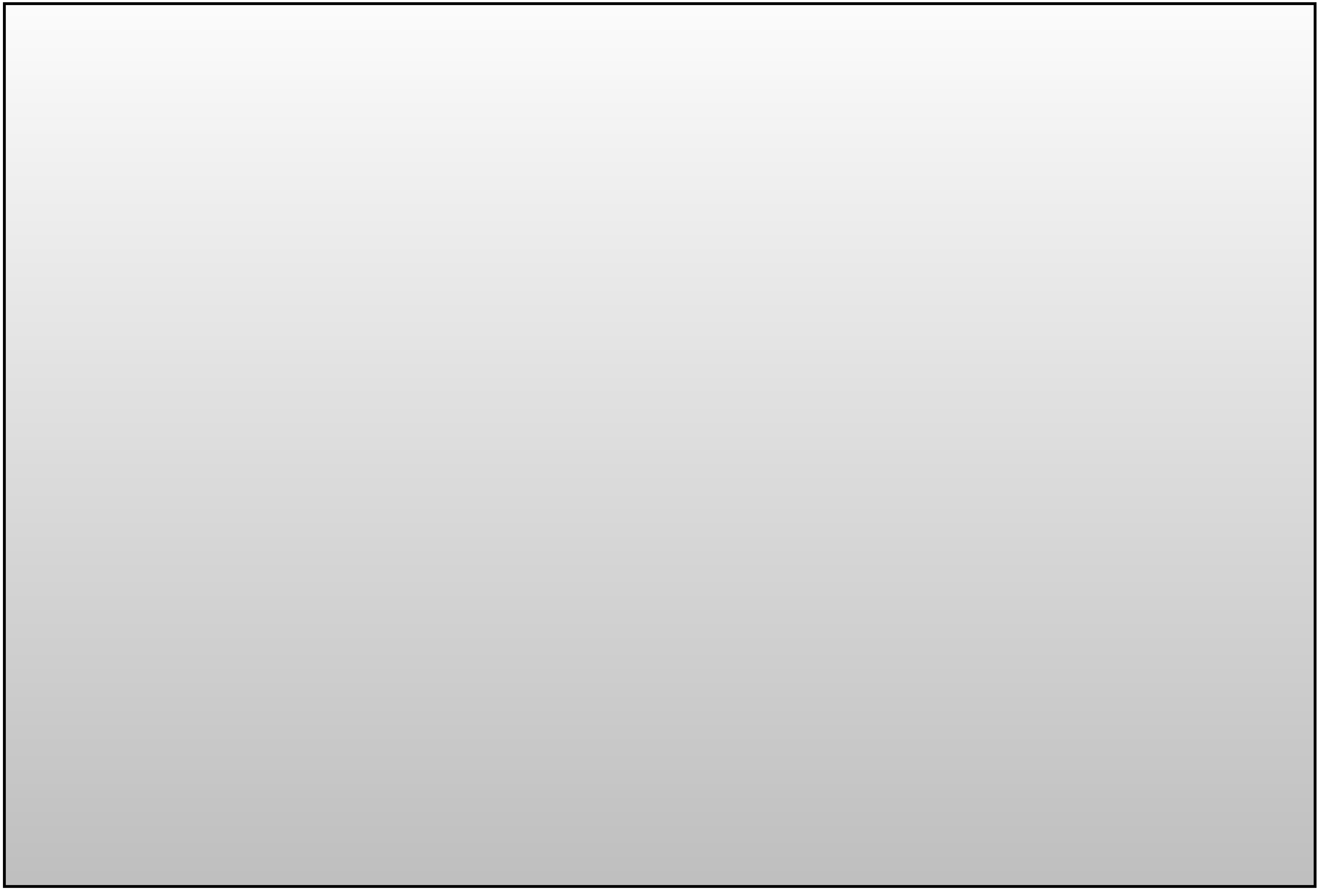
$\mathbf{e}_{ij}^{+}$
$\mathbf{v}_i^{+}$
$\mathbf{v}_j^{+}$
$2\mathbf{n}_i$
$\mathbf{e}_{ij}$
$2\mathbf{n}_j$
$\mathbf{v}_i^{-}$
$\mathbf{e}_{ij}^{-}$
$\mathbf{v}_j^{-}$

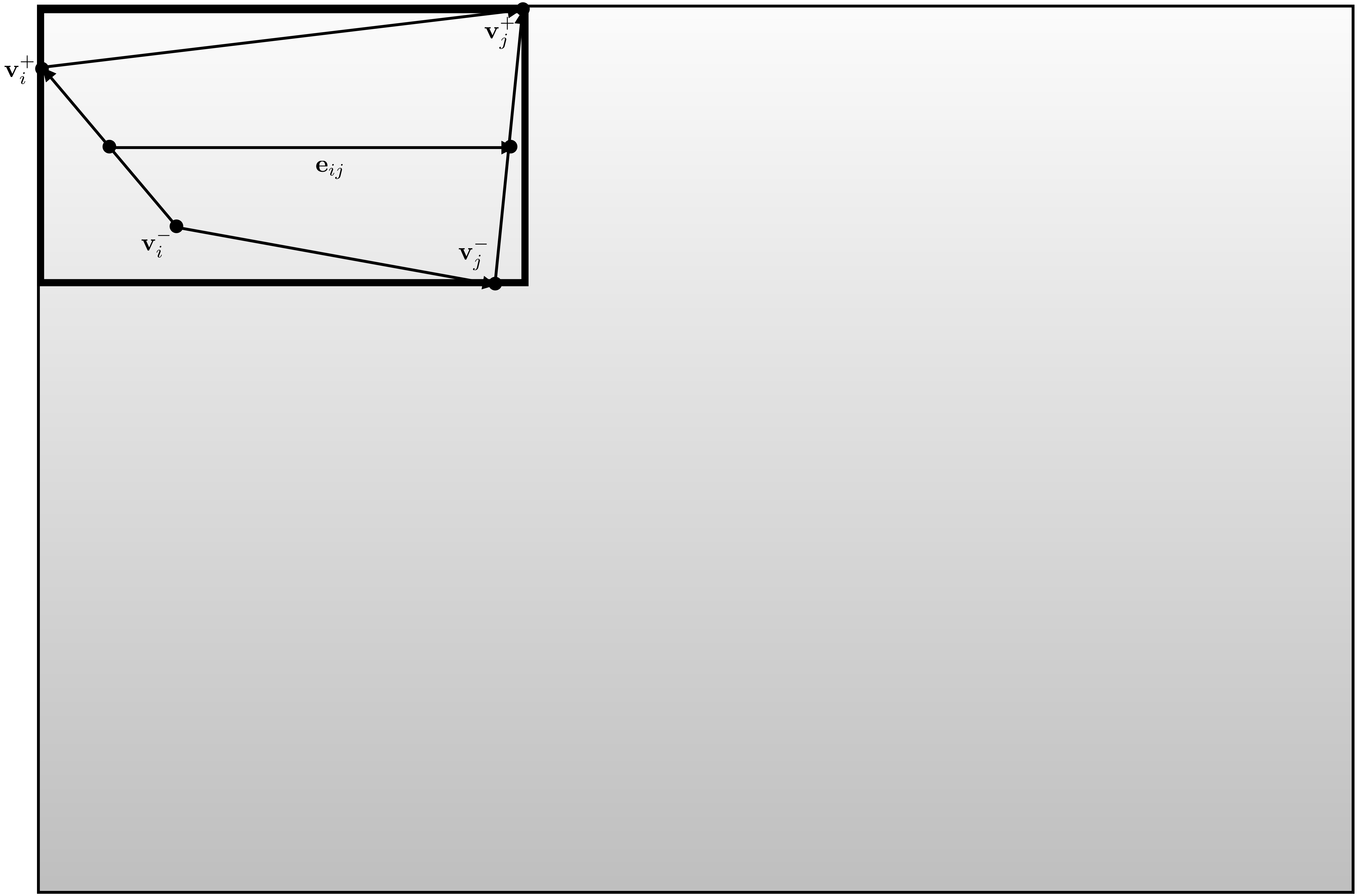


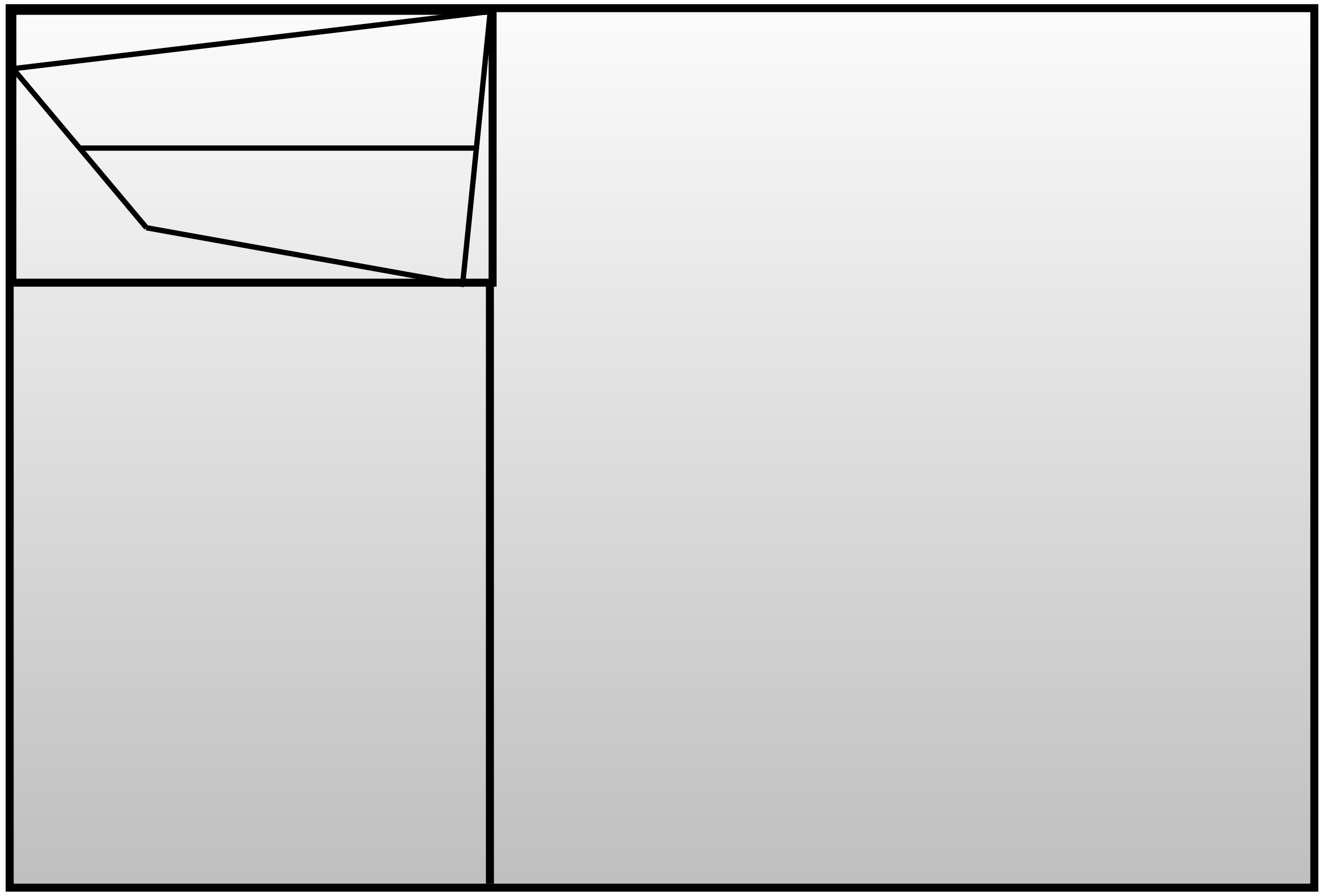
Cut Plan

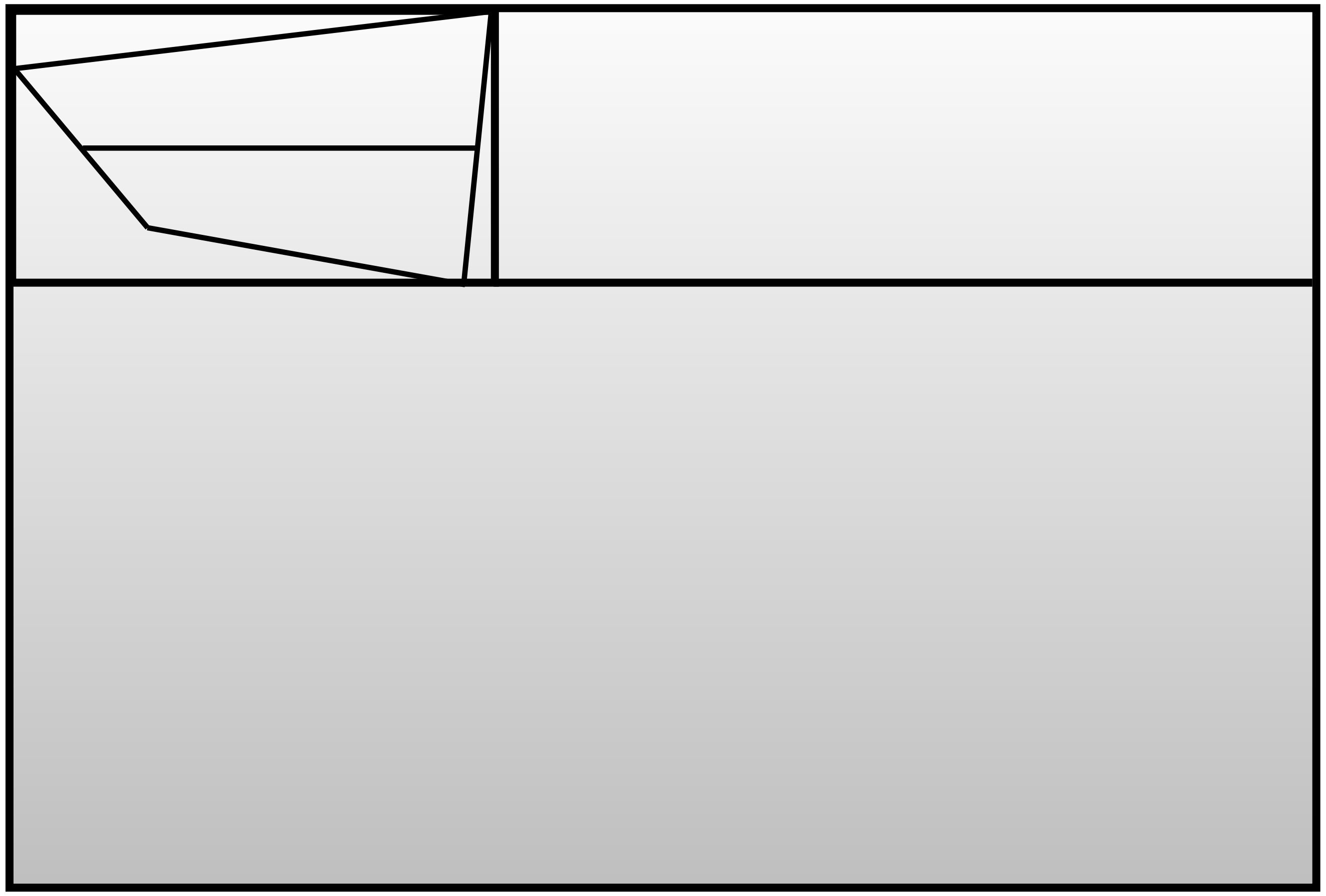

$\mathbf{v}_j^+$
$\mathbf{v}_i^+$
$\mathbf{e}_{ij}$
$\mathbf{v}_i^-$
$\mathbf{v}_j^-$

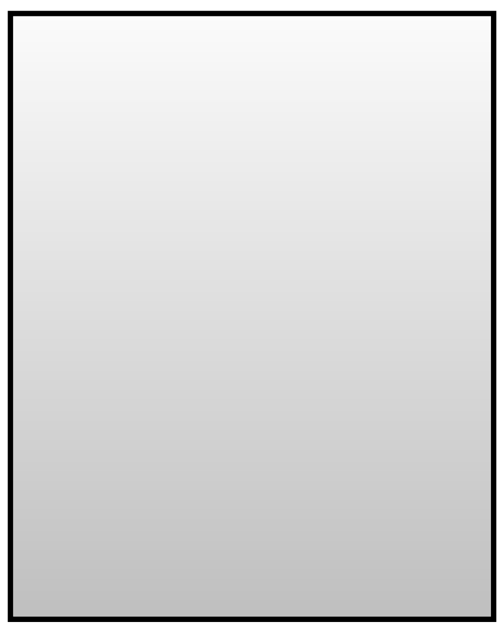
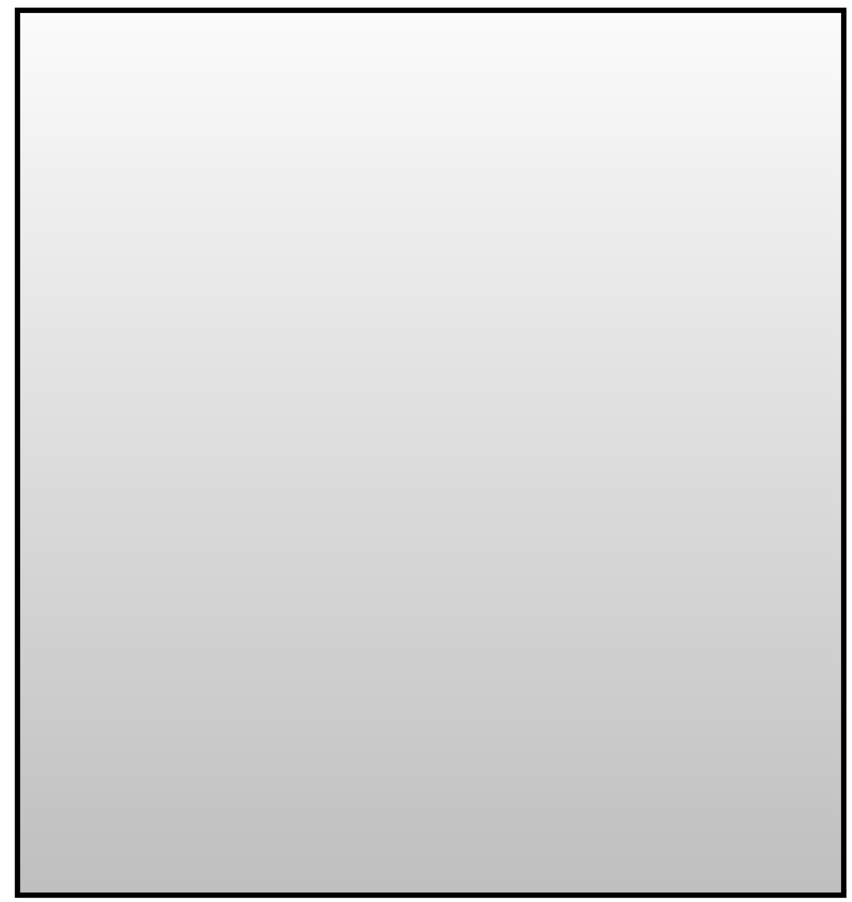


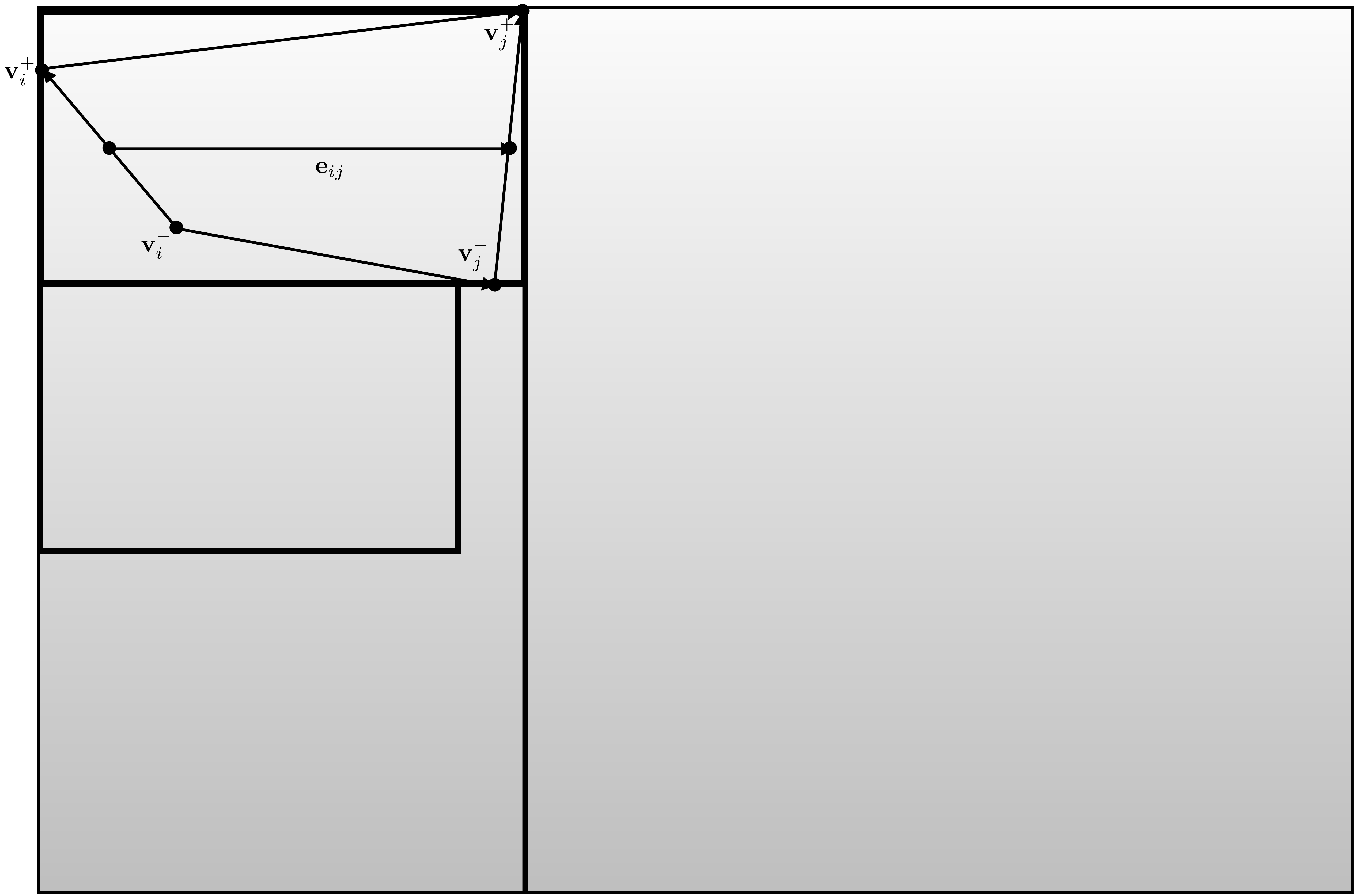



$\mathbf{v}_i^+$
$\mathbf{v}_j^+$
$\mathbf{e}_{ij}$
$\mathbf{v}_i^-$
$\mathbf{v}_j^-$





Q =  , 



$\mathbf{v}_i^+$
$\mathbf{v}_j^+$
$\mathbf{e}_{ij}$
$\mathbf{v}_i^-$
$\mathbf{v}_j^-$