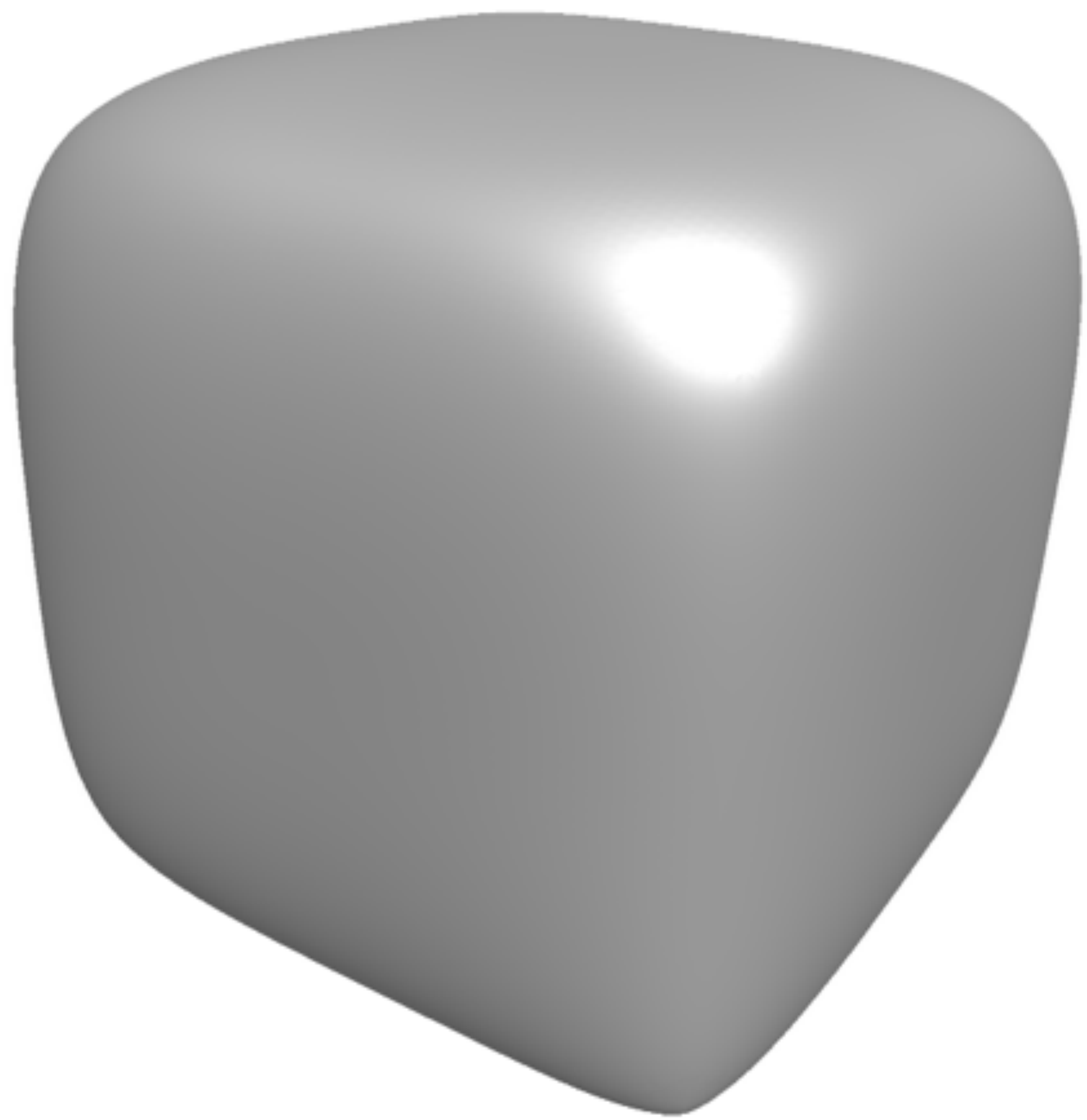
Mahalanobis Centroidal Voronoi Tessellations

Ronald Richter, Marc Alexa
TU Berlin

Motivation



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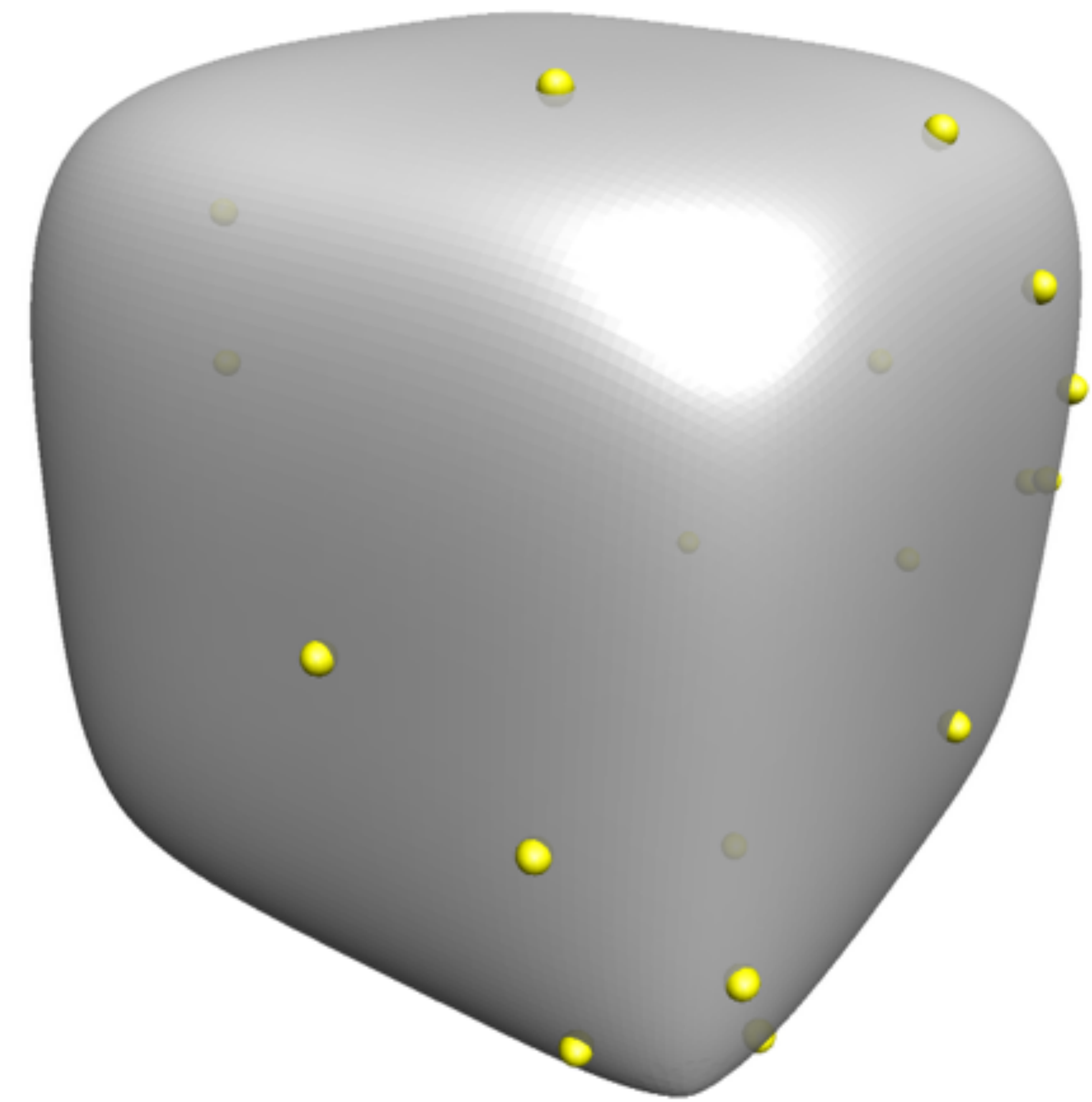

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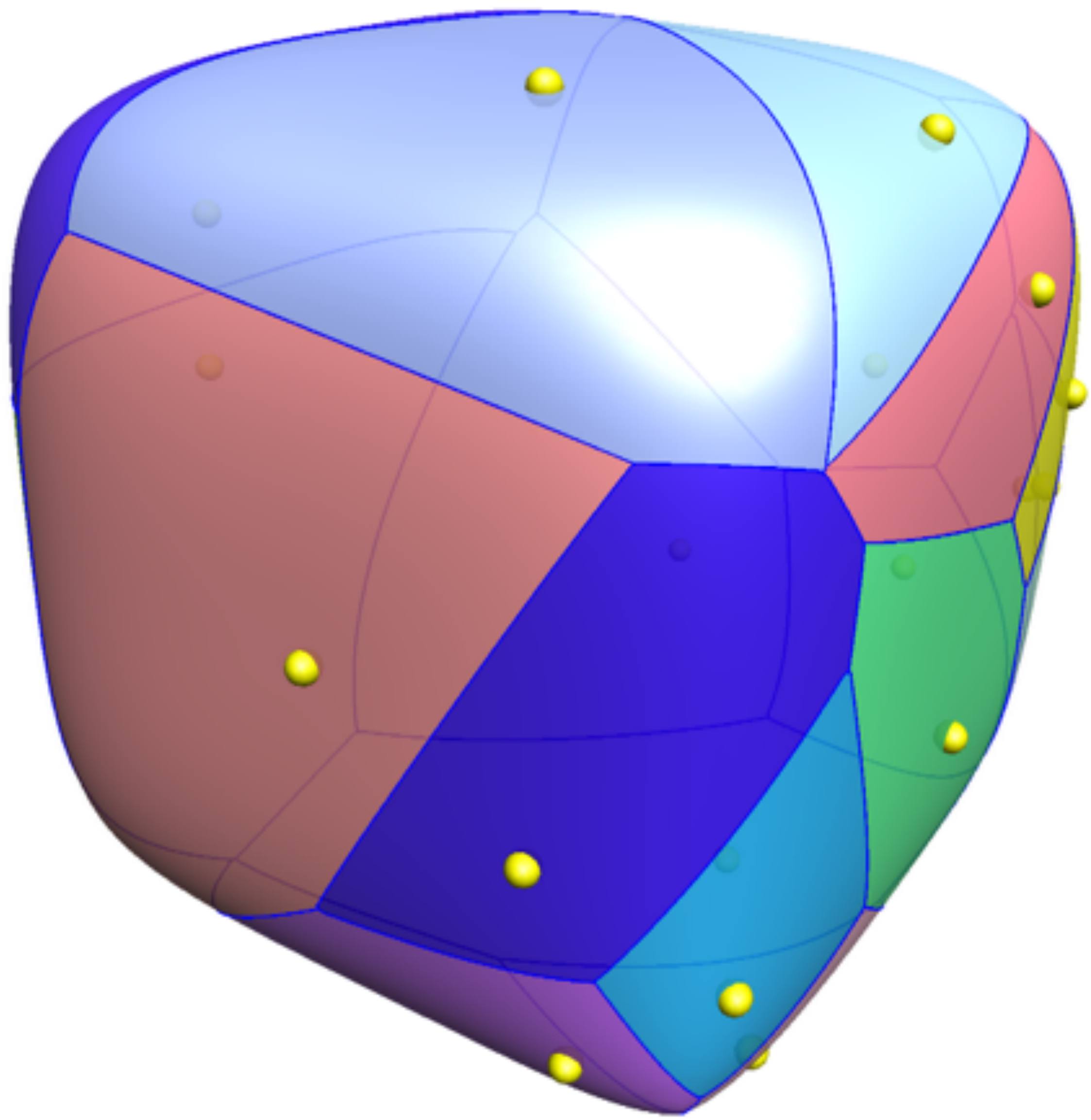

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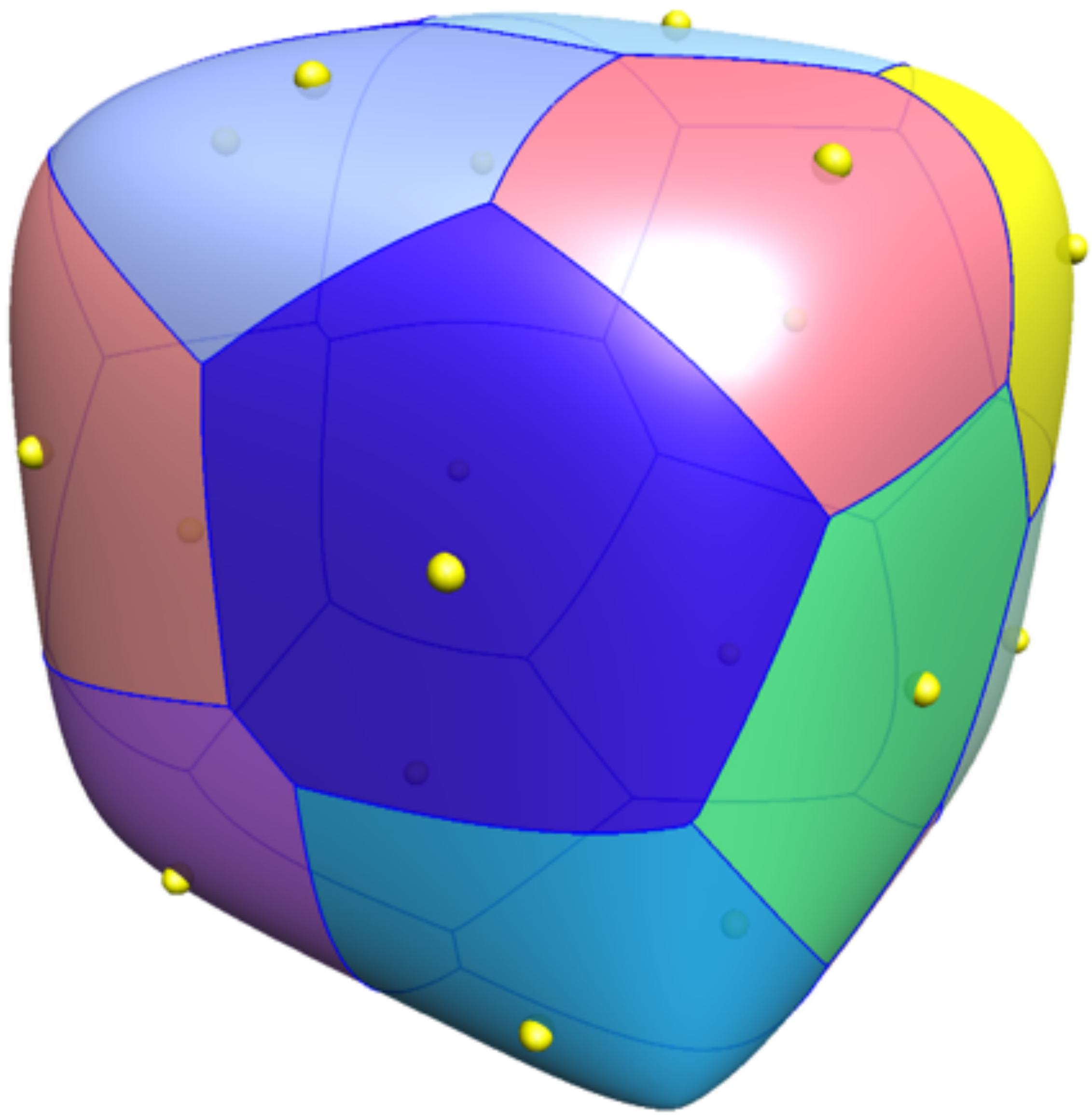

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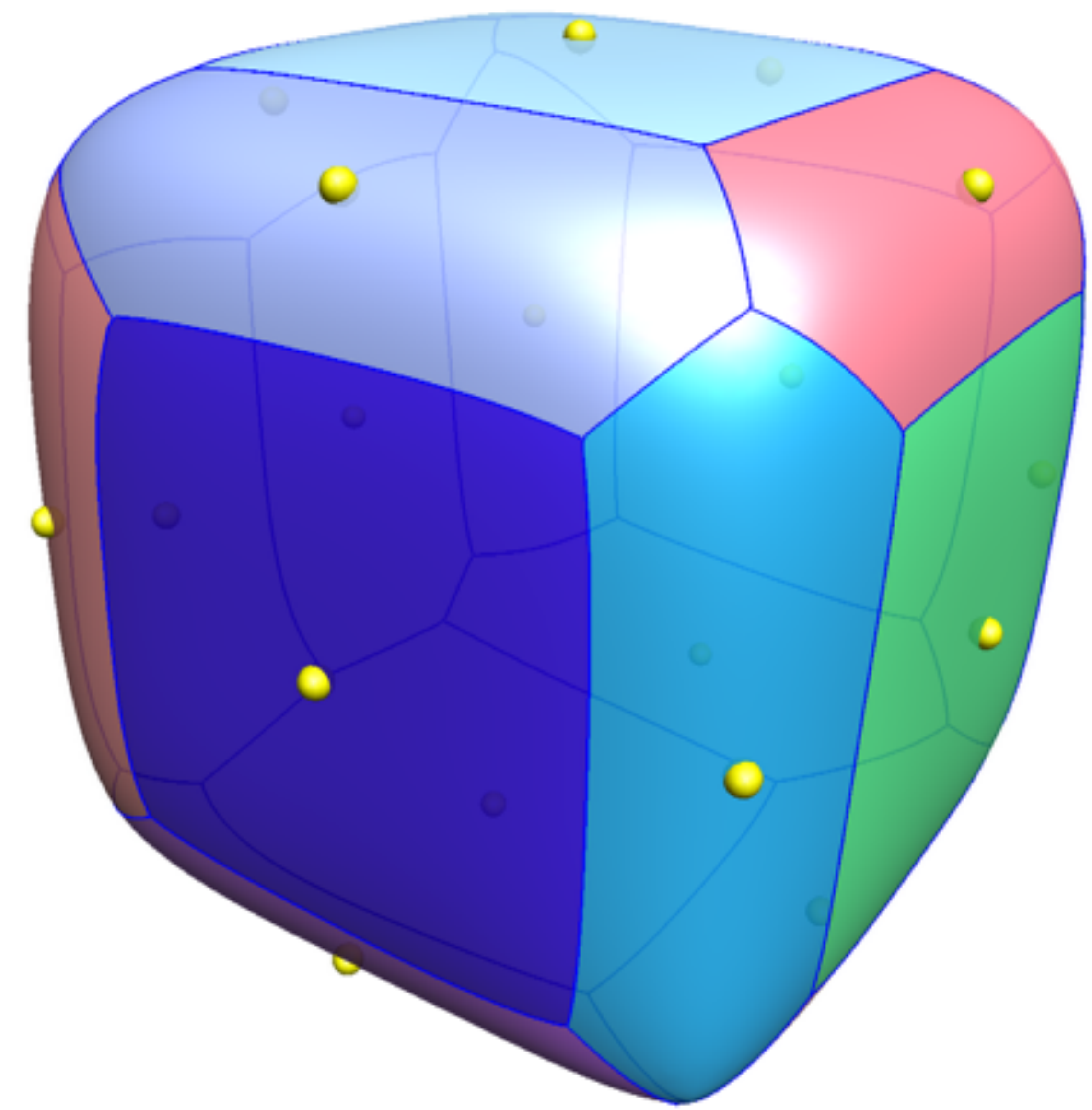

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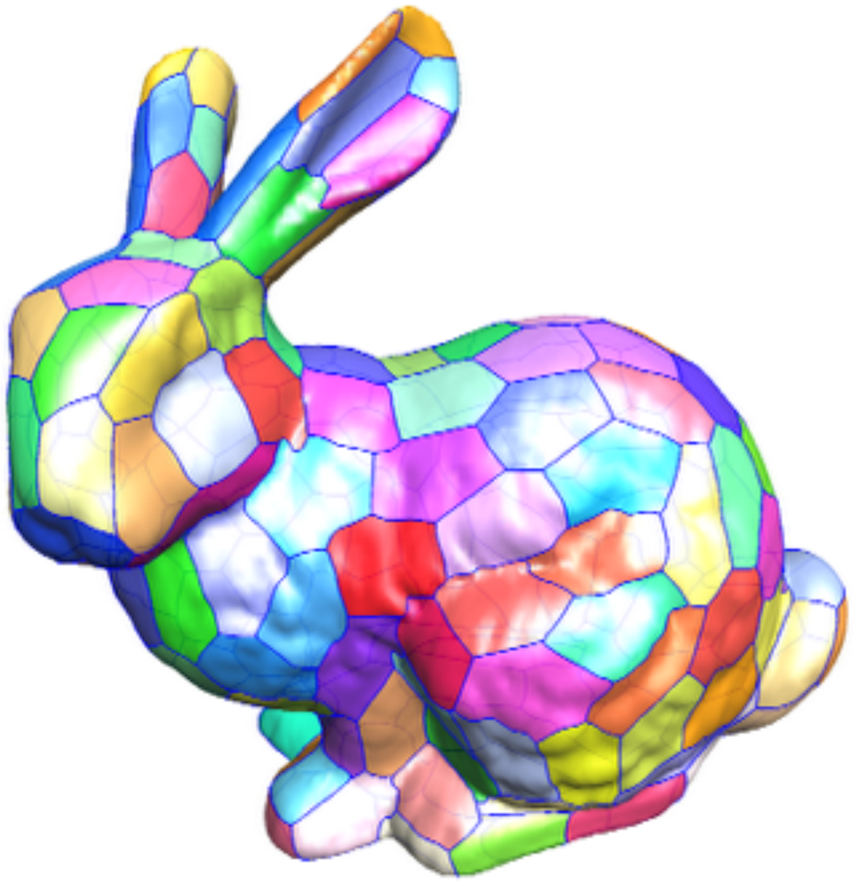
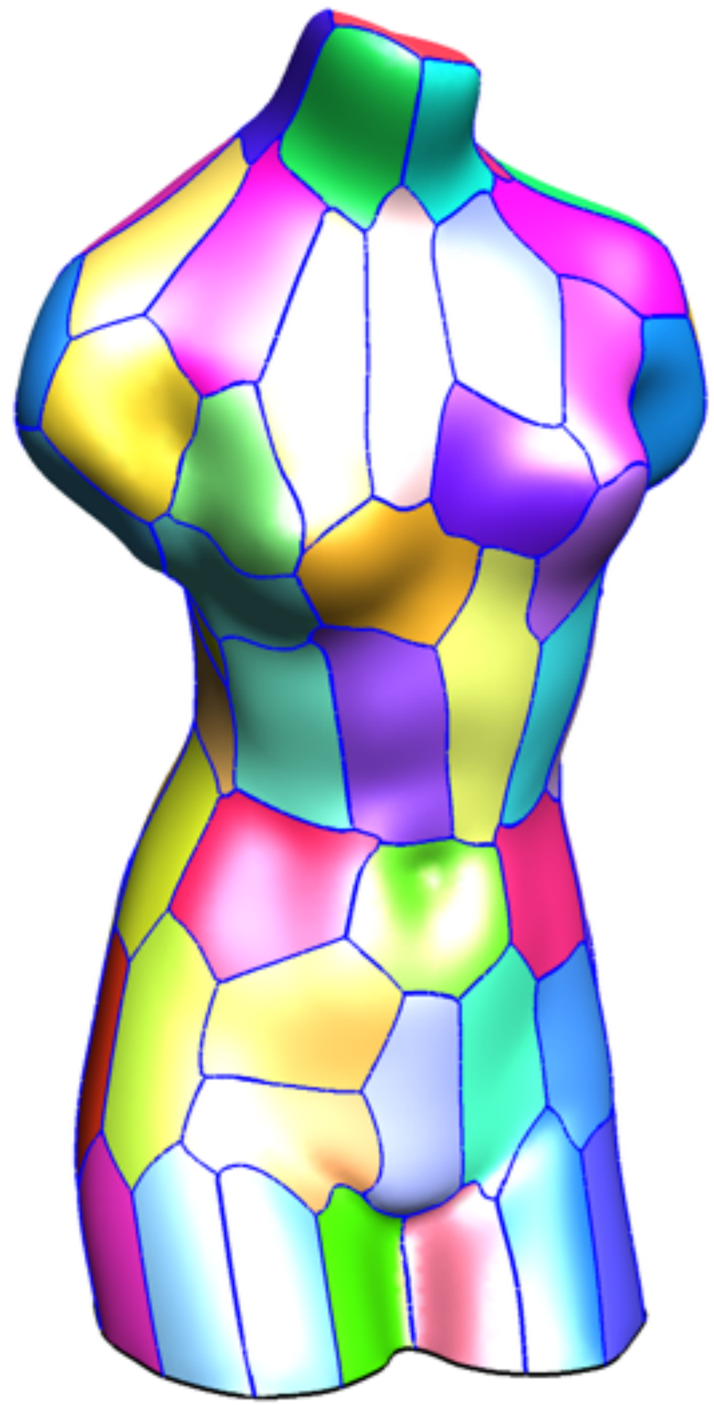
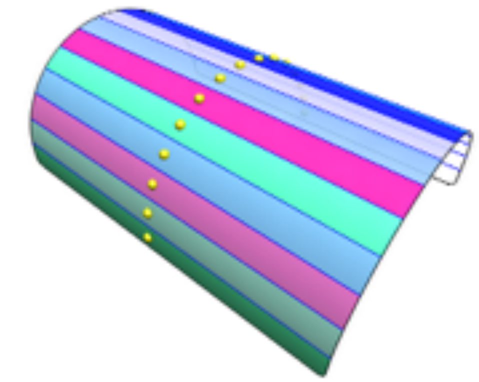
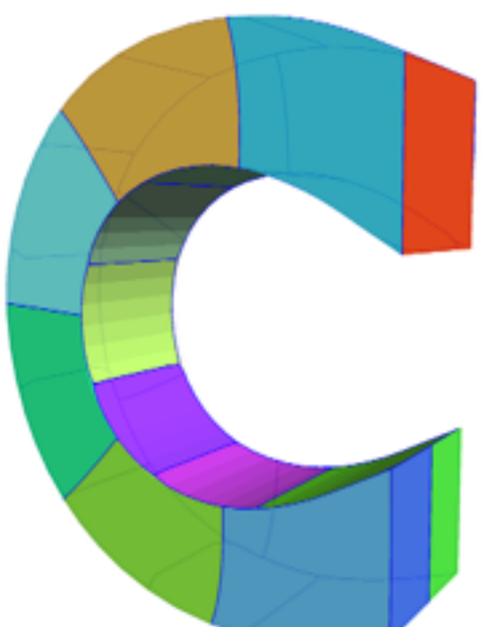
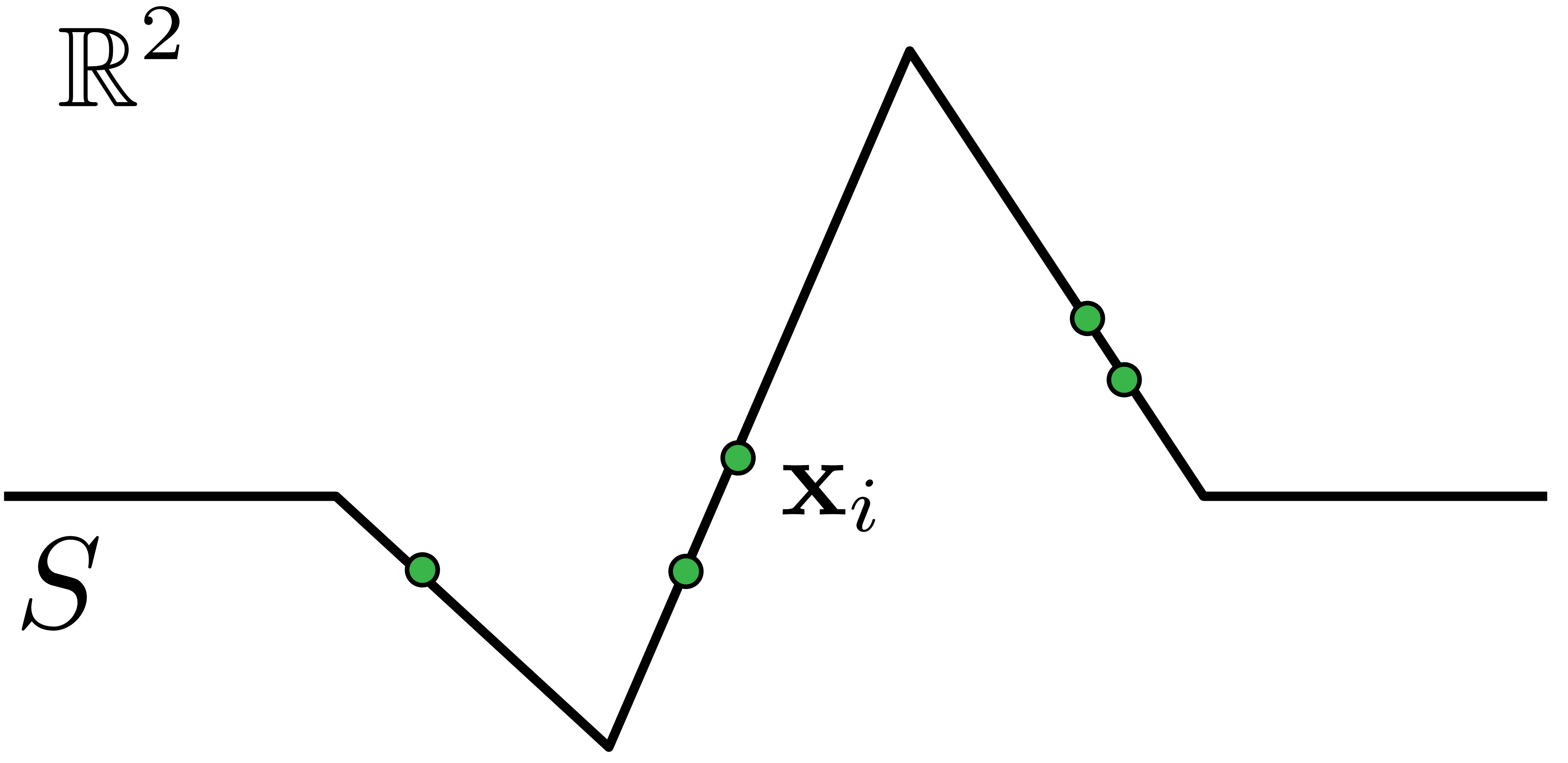




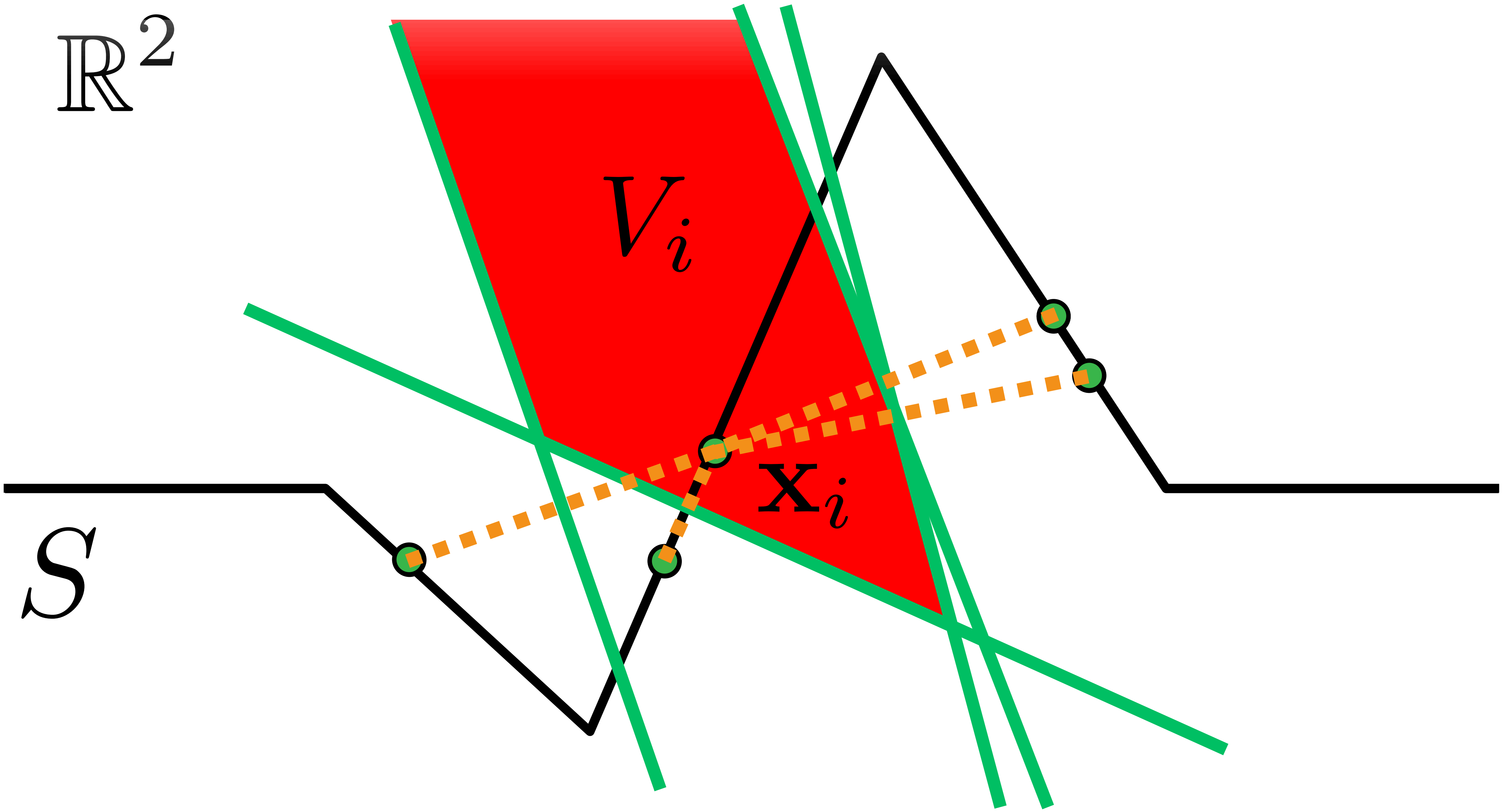




Setup

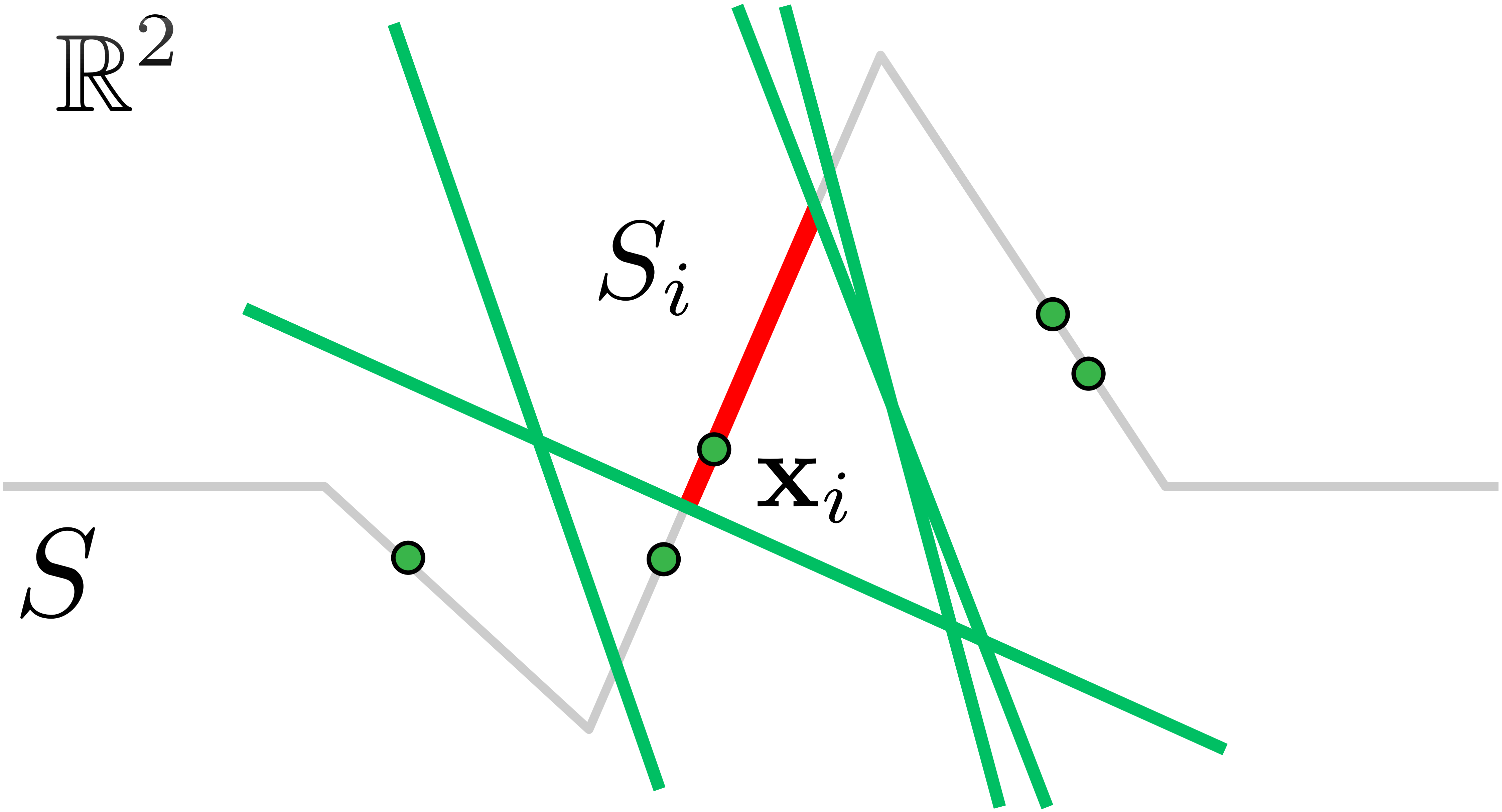


Voronoi tessellation



$$V_i = \{\mathbf{p} : d(\mathbf{x}_i, \mathbf{p}) < d(\mathbf{x}_j, \mathbf{p}), \forall j \neq i\} \quad \mathbf{p} \in \mathbb{R}^2$$

Setup



$$S_i = \{\mathbf{p} : d(\mathbf{x}_i, \mathbf{p}) < d(\mathbf{x}_j, \mathbf{p}), \forall j \neq i\} \quad \mathbf{p} \in S$$

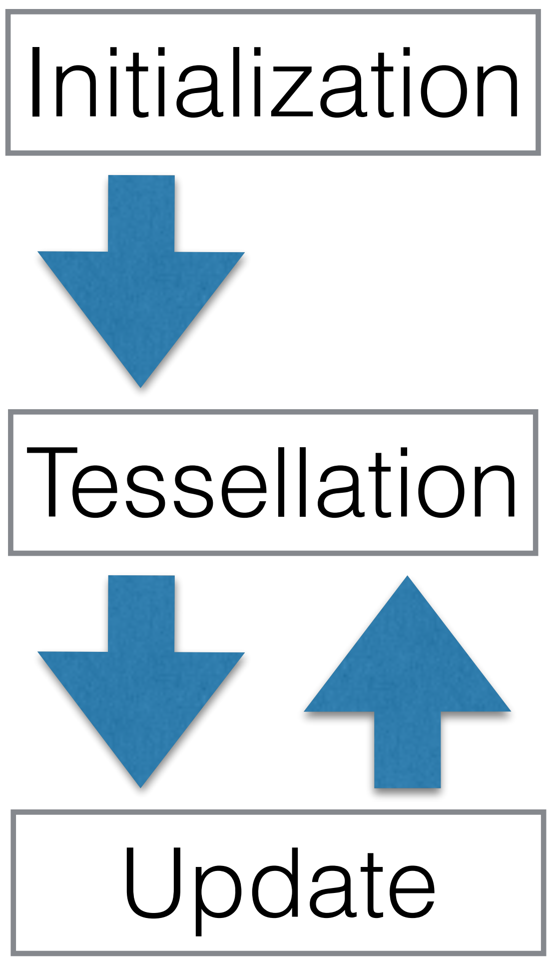
CVT

$$F(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}) = \sum_i \int_{S_i} d^2(\mathbf{x}_i, \mathbf{p}) d\mathbf{p}$$

$$\frac{\partial F}{\partial \mathbf{x}_i} = \int_{S_i} 2(\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p}$$

$$\mathbf{x}_i^{new} = \frac{\int_{S_i} \mathbf{p}\, d\mathbf{p}}{\int_{S_i} 1\, d\mathbf{p}}$$

Algorithm (Lloyd)



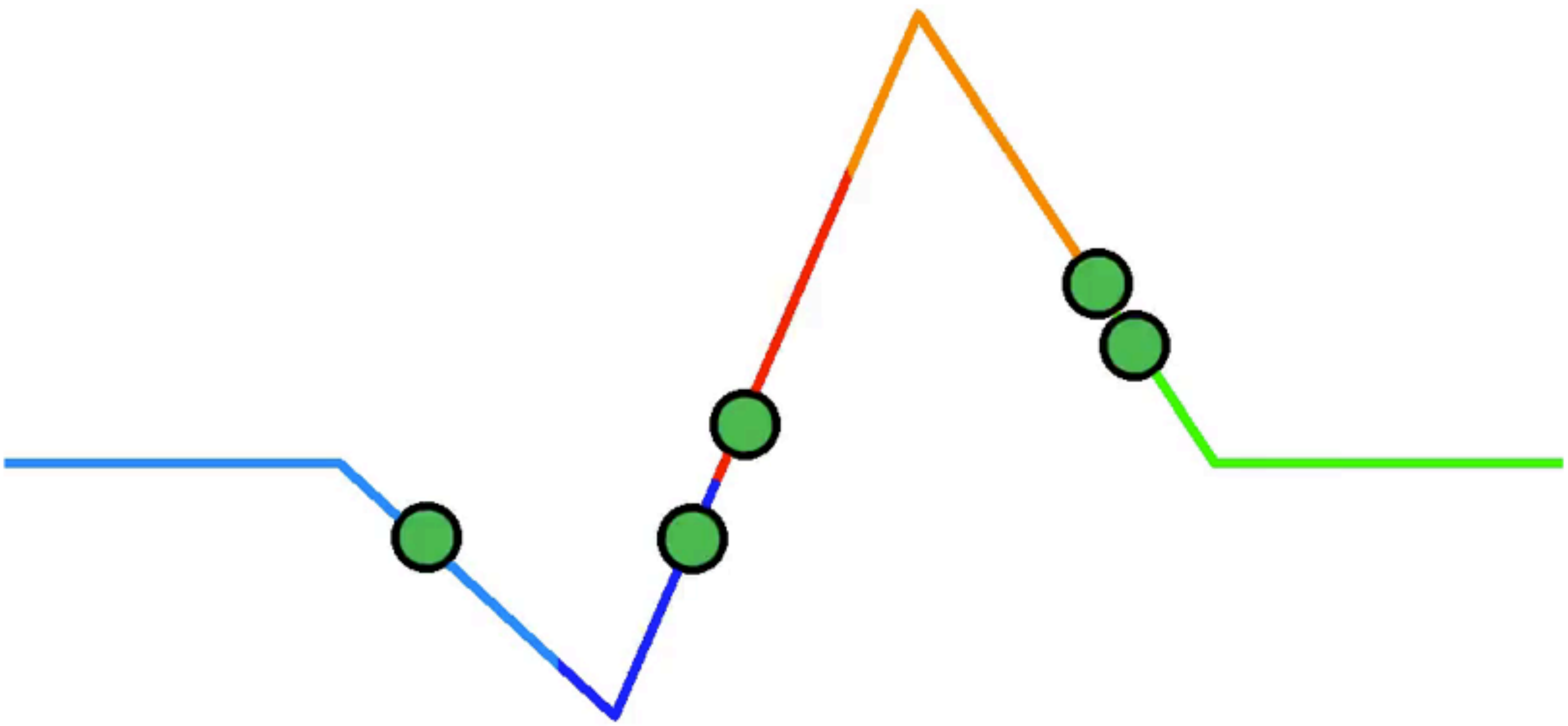
$\mathbf{x}_i = rand(S)$

FIXED $\mathbf{x}_i$
COMPUTE S_i

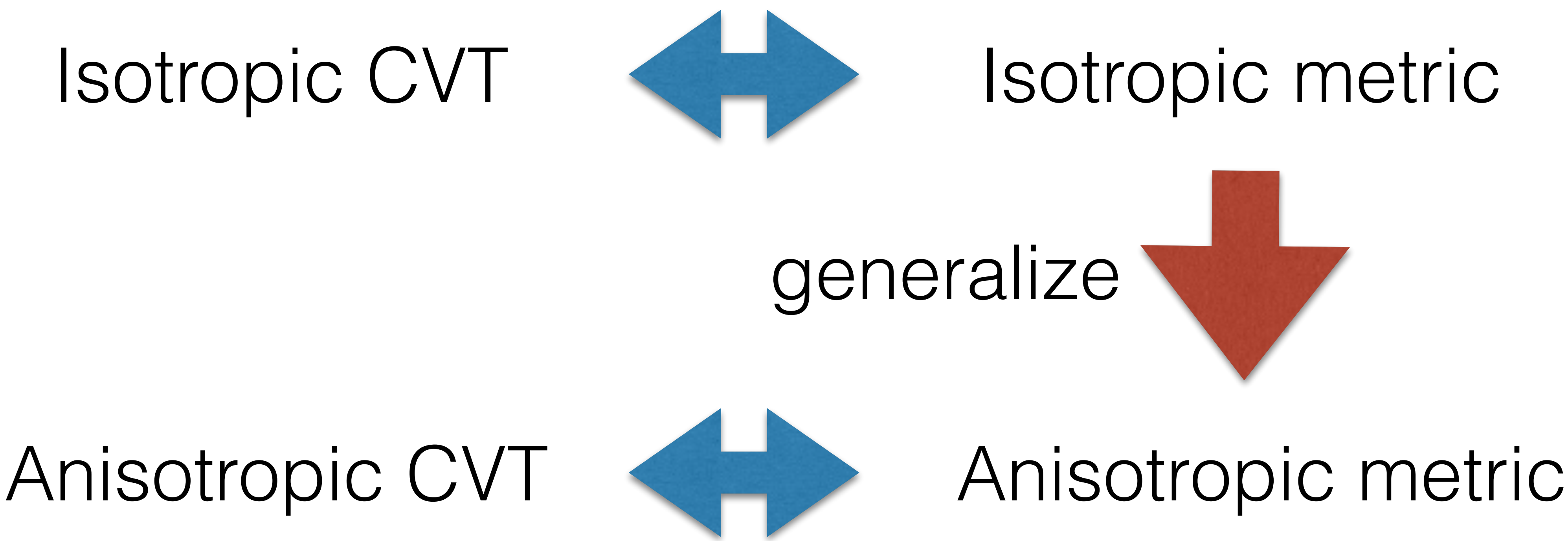
FIXED S_i
UPDATE $\mathbf{x}_i$

CVT

1



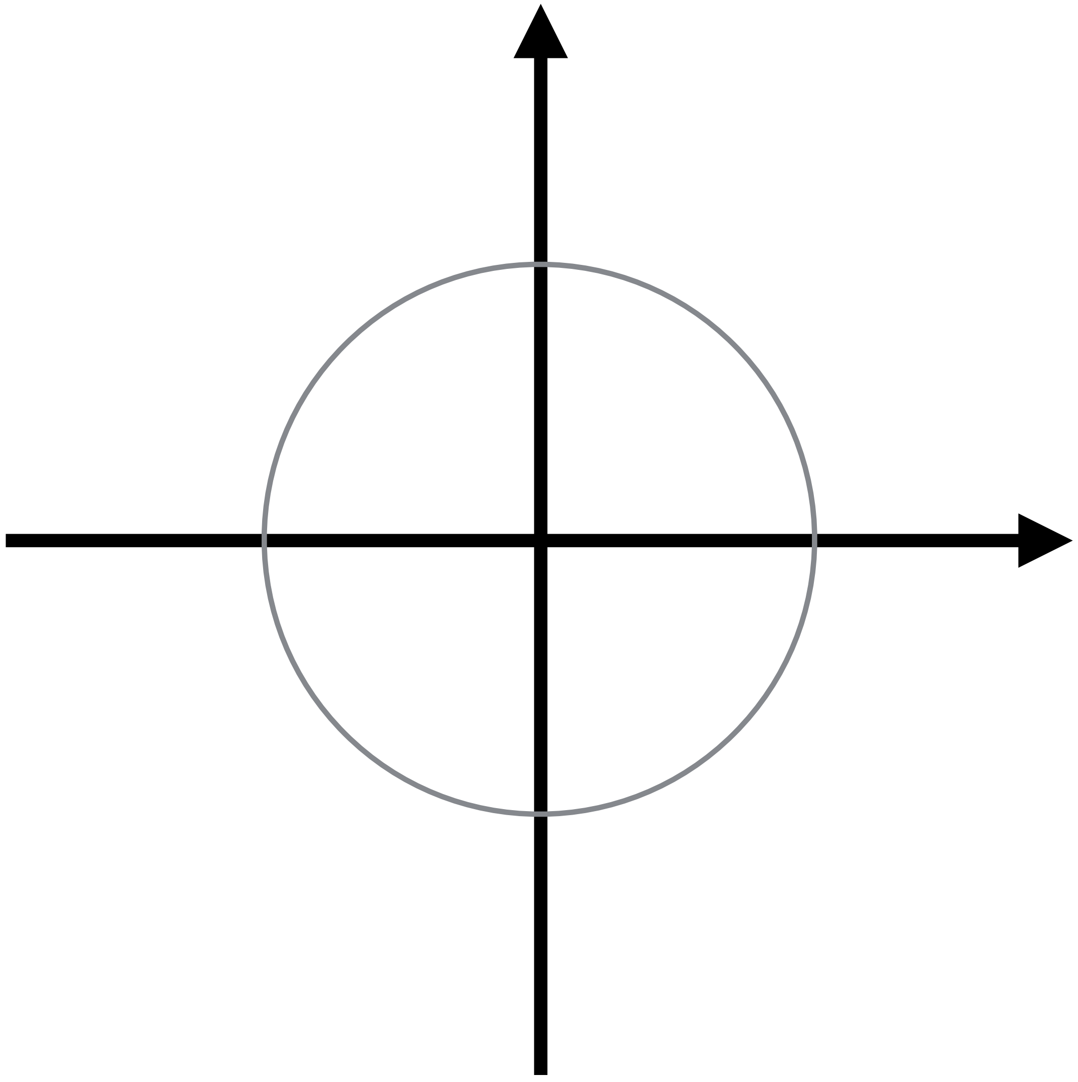
Metric


Isotropic CVT
Isotropic metric
generalize
Anisotropic CVT
Anisotropic metric

Isotropic metric

$$d(\mathbf{x}, \mathbf{p})^2 = (\mathbf{x} - \mathbf{p})^\mathsf{T} \mathbf{M} (\mathbf{x} - \mathbf{p})$$

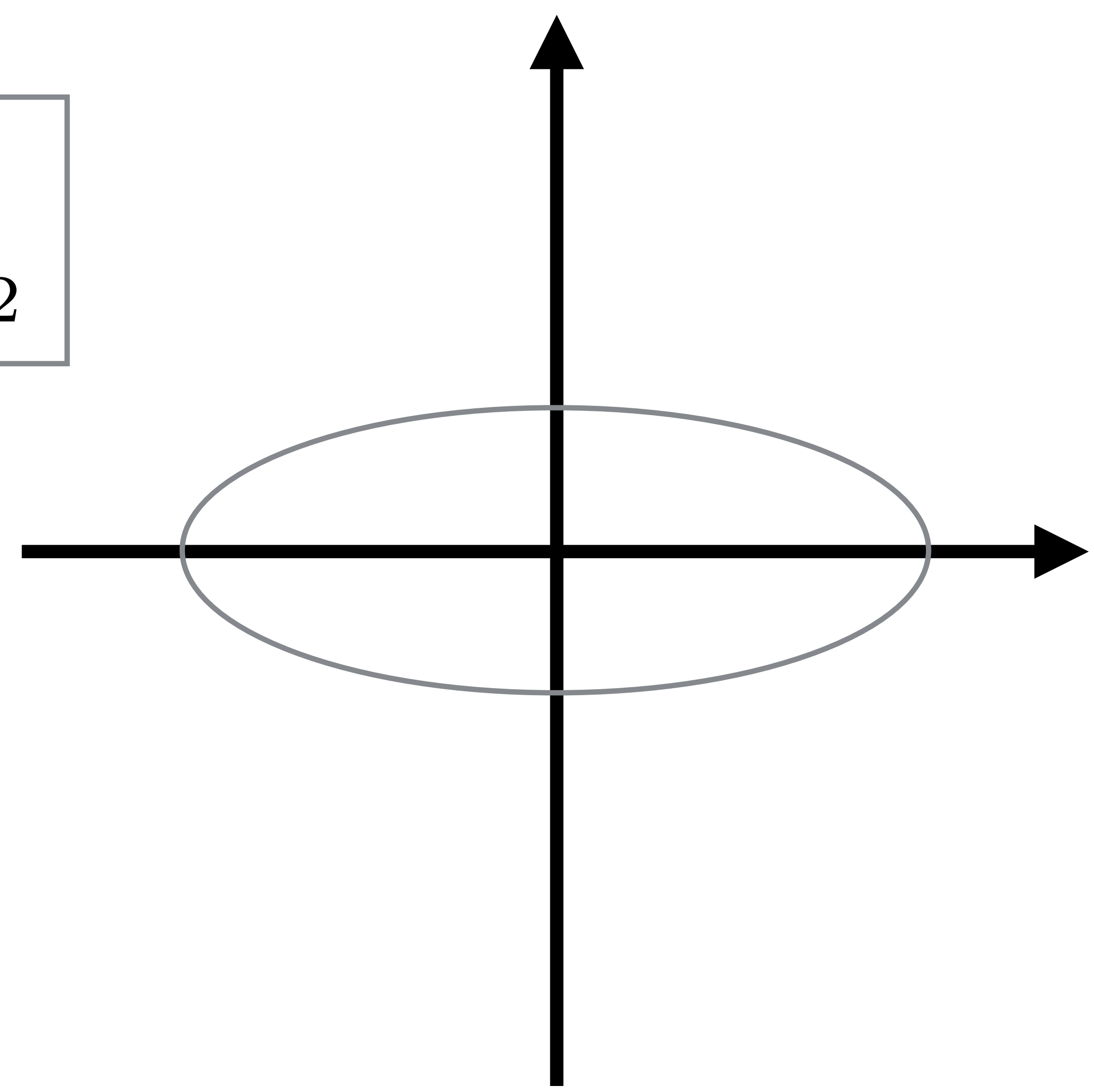
$$\mathbf{M} = \mathbf{I}$$



Anisotropic metric

$$d(\mathbf{x},\mathbf{p})^2 = (\mathbf{x}-\mathbf{p})^\mathsf{T}\mathbf{M}(\mathbf{x}-\mathbf{p})$$

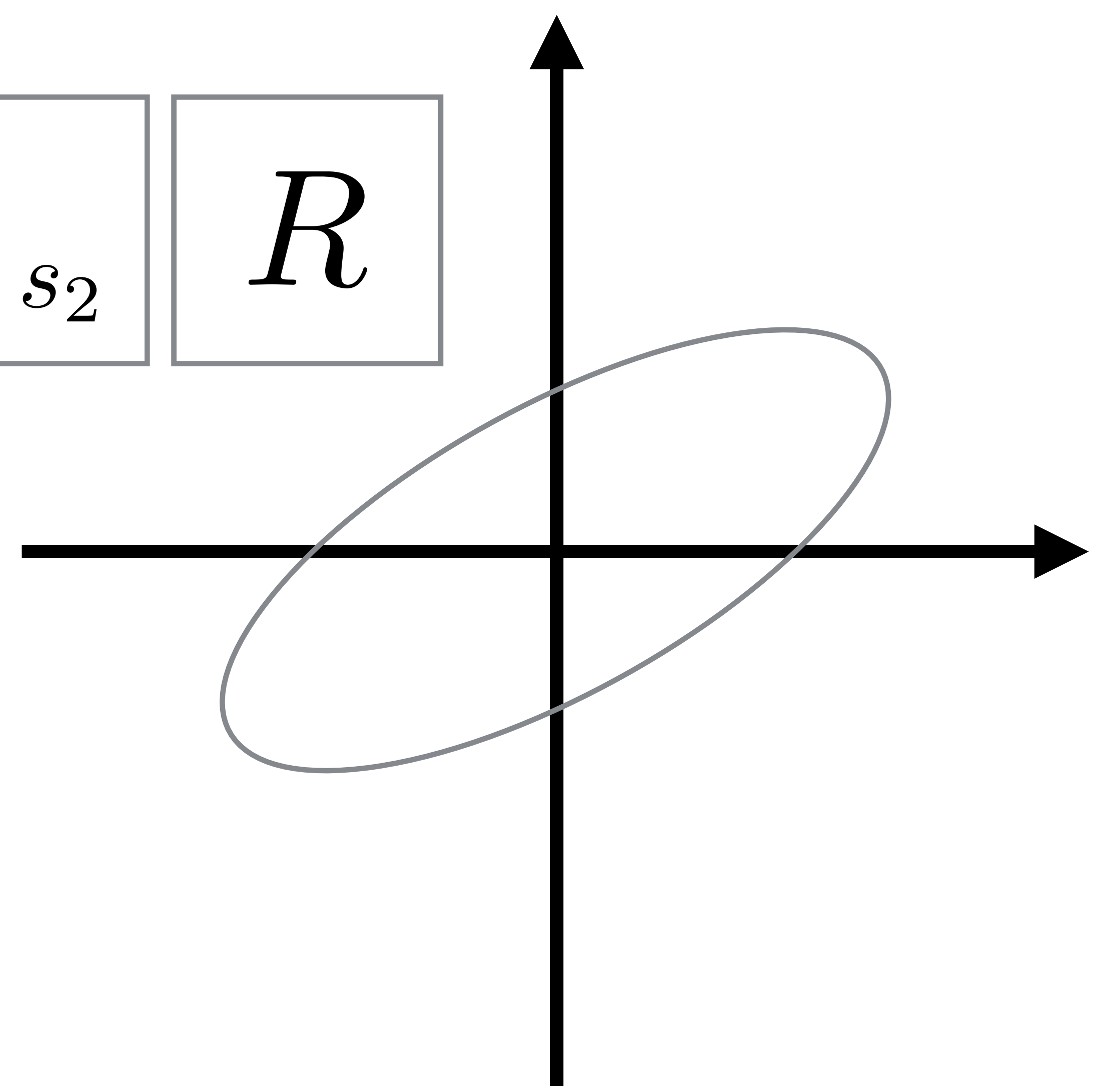
$$\mathbf{M} = \begin{bmatrix} s_1 & \\ & s_2 \end{bmatrix}$$



Anisotropic metric

$$d(\mathbf{x}, \mathbf{p})^2 = (\mathbf{x} - \mathbf{p})^{\mathsf{T}} \mathbf{M} (\mathbf{x} - \mathbf{p})$$

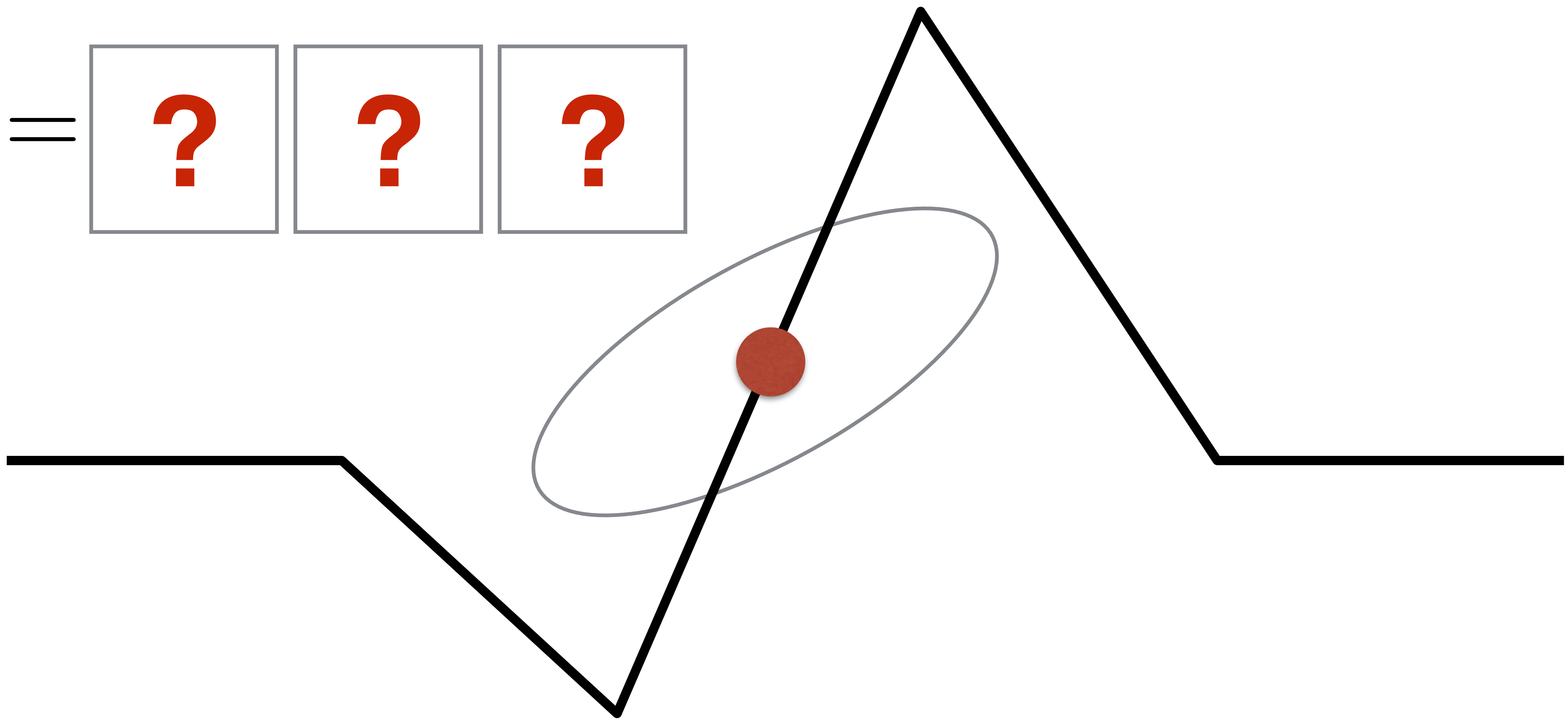
$$\mathbf{M} = \boxed{R^{-1}} \boxed{\begin{matrix} s_1 & \\ & s_2 \end{matrix}} \boxed{R}$$



Anisotropic metric

$$d(\mathbf{x}, \mathbf{p})^2 = (\mathbf{x} - \mathbf{p})^\top \mathbf{M} (\mathbf{x} - \mathbf{p})$$

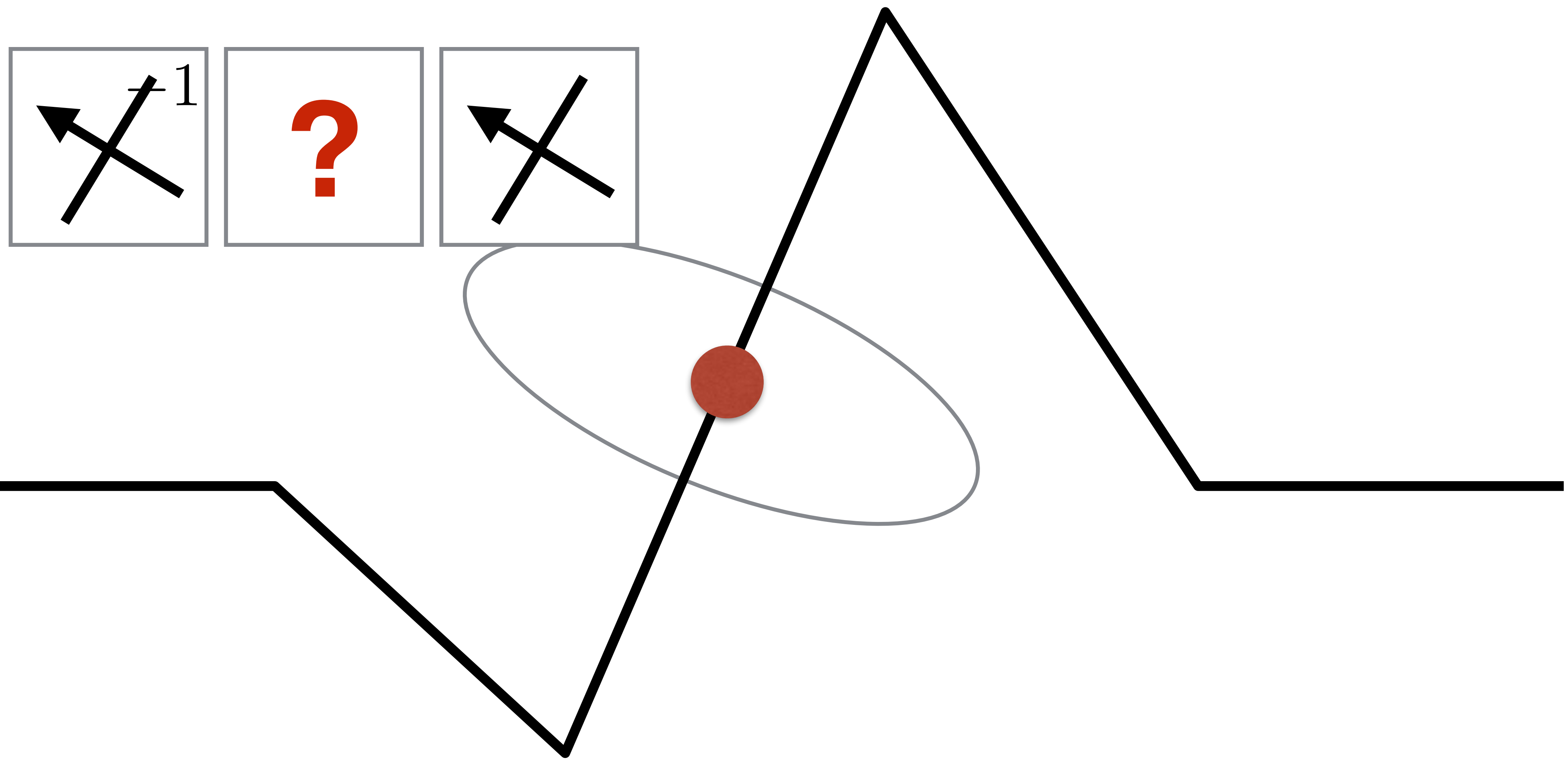
$\mathbf{M} =$ | ? | ? | ? |



Anisotropic metric

$$d(\mathbf{x}, \mathbf{p})^2 = (\mathbf{x} - \mathbf{p})^\mathsf{T} \mathbf{M} (\mathbf{x} - \mathbf{p})$$

$\mathbf{M} =$



Anisotropic CVT

Minimize:

$$F(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{M}_0, \ldots, \mathbf{M}_{k-1}) =$$

$$\sum_i \int_{S_i} (\mathbf{p} - \mathbf{x}_i)^\mathsf{T} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p}$$

Problem: trivial solutions

$$\mathbf{M}_i = \mathbf{0}$$

Solution: add constraint

$|\mathbf{M}_i| = 1$ (other constraints possible)

Mahalanobis CVT

Minimize:

$$L(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{M}_0, \ldots, \mathbf{M}_{k-1}) =$$

$$\sum_i \left[\int_{S_i} (\mathbf{p} - \mathbf{x}_i)^\mathsf{T} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \;\; + \;\; \lambda \left(|\mathbf{M}_i| - 1 \right) \right]$$

CVT energy

Lagrange multiplier constraining M

Mahalanobis CVT

$$\sum_i \left[\int_{S_i} (\mathbf{p} - \mathbf{x}_i)^{\mathsf{T}} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \quad + \quad \lambda \left(|\mathbf{M}_i| - 1 \right) \right]$$

$$\frac{\partial L}{\partial \mathbf{x}_i} : \qquad \int_{S_i} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \qquad\qquad \mathbf{0}$$

$$\mathbf{x}_i^{new} = \frac{\int_{S_i} \mathbf{p}\, d\mathbf{p}}{\int_{S_i} 1\, d\mathbf{p}}$$

independent of M
(as for isotropic CVT)

Mahalanobis CVT

$$\sum_i \left[\int_{S_i} (\mathbf{p} - \mathbf{x}_i)^{\mathsf{T}} \mathbf{M}_i (\mathbf{p} - \mathbf{x}_i)\, d\mathbf{p} \;+\; \lambda\,(|\mathbf{M}_i| - 1) \right]$$

$$\frac{\partial L}{\partial \mathbf{M}_i} : \quad \int_{S_i} (\mathbf{p} - \mathbf{x}_i)(\mathbf{p} - \mathbf{x}_i)^T d\mathbf{p} \qquad \lambda |\mathbf{M}_i| \; \left(\mathbf{M}_i^{-1}\right)^{\mathsf{T}}$$

$= \mathbf{C}_i$ *covariance matrix*

$$\mathbf{C}_i + \lambda |\mathbf{M}_i| \; \left(\mathbf{M}_i^{-1}\right)^{\mathsf{T}} = \mathbf{0} \quad \Rightarrow \quad \mathbf{M}_i^{new} = |\mathbf{C}_i|^{\frac{1}{n}} \mathbf{C}_i^{-1}$$

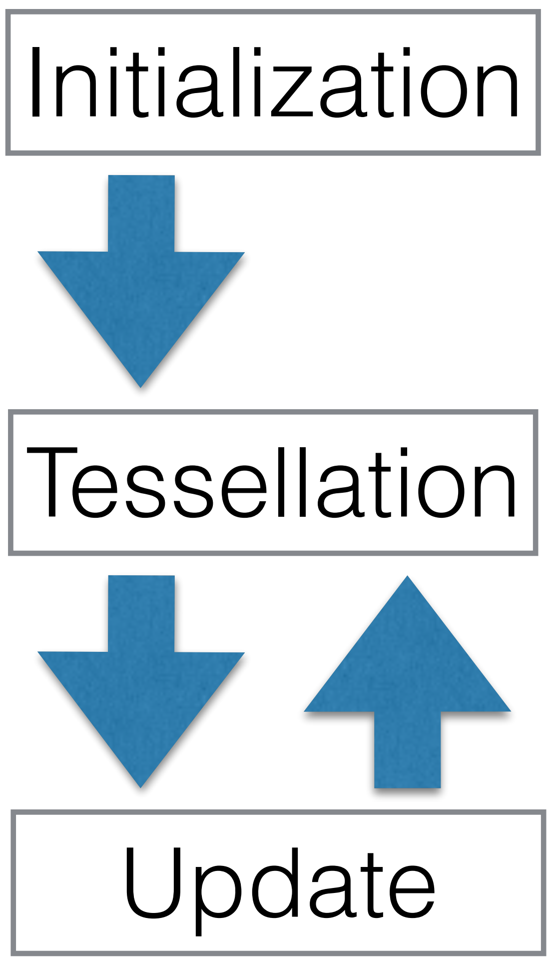
Mahalanobis distance

$$d(\mathbf{x}_i, \mathbf{p})^2 = (\mathbf{x}_i - \mathbf{p})^\mathsf{T} \mathbf{M}_i (\mathbf{x}_i - \mathbf{p})$$

$$d(\mathbf{x}_i, \mathbf{p})^2 = |\mathbf{C}|^{\frac{1}{n}} (\mathbf{x}_i - \mathbf{p})^\mathsf{T} \mathbf{C}^{-1} (\mathbf{x}_i - \mathbf{p})$$

Mahalanobis distance (squared)

Algorithm (Lloyd)



$\mathbf{x}_i = rand(S)$

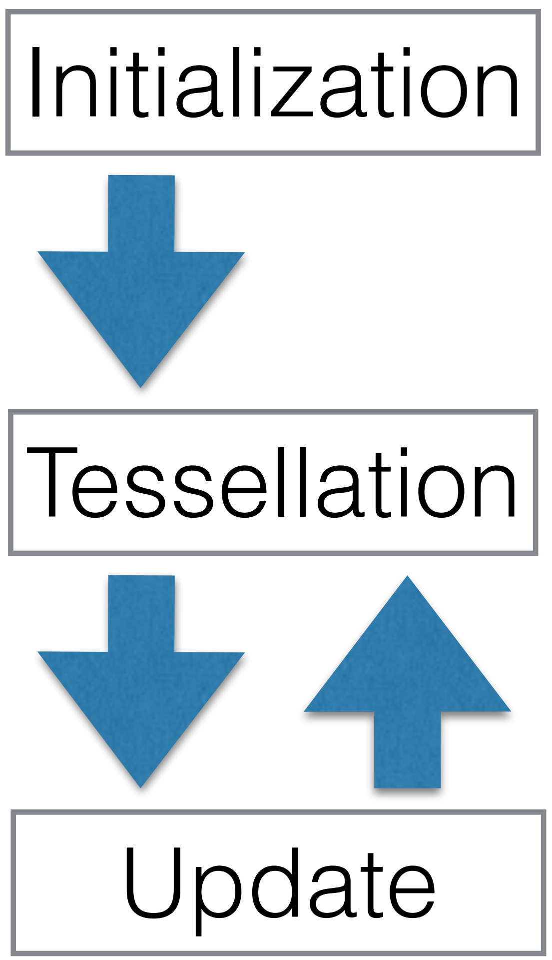
FIXED $\mathbf{x}_i$

COMPUTE S_i

FIXED S_i

UPDATE $\mathbf{x}_i$

Algorithm



$\mathbf{x}_i = rand(S)$

$\mathbf{M}_i = \mathbf{I}$

FIXED $\mathbf{x}_i, \mathbf{M}_i$

COMPUTE S_i

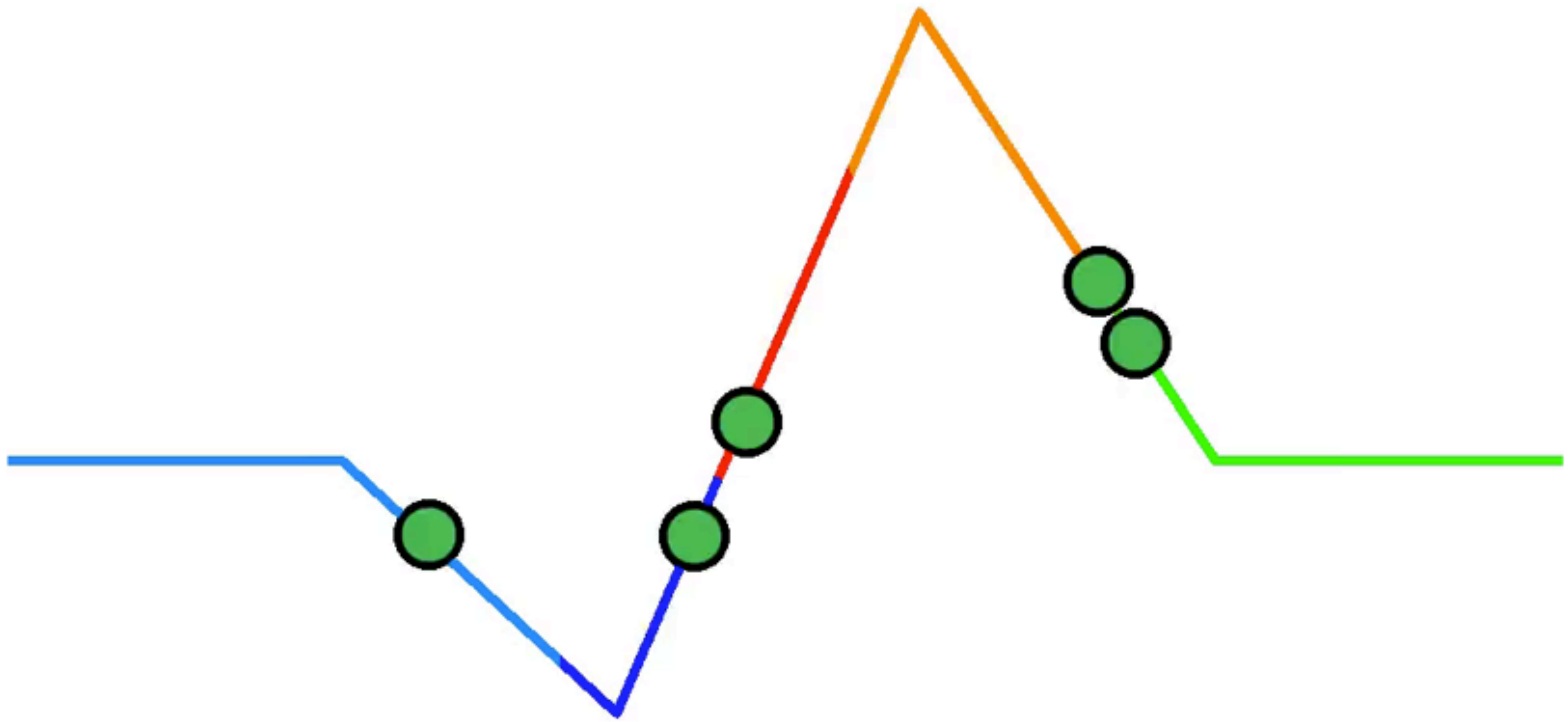
FIXED S_i

UPDATE $\mathbf{x}_i$

UPDATE $\mathbf{M}_i$

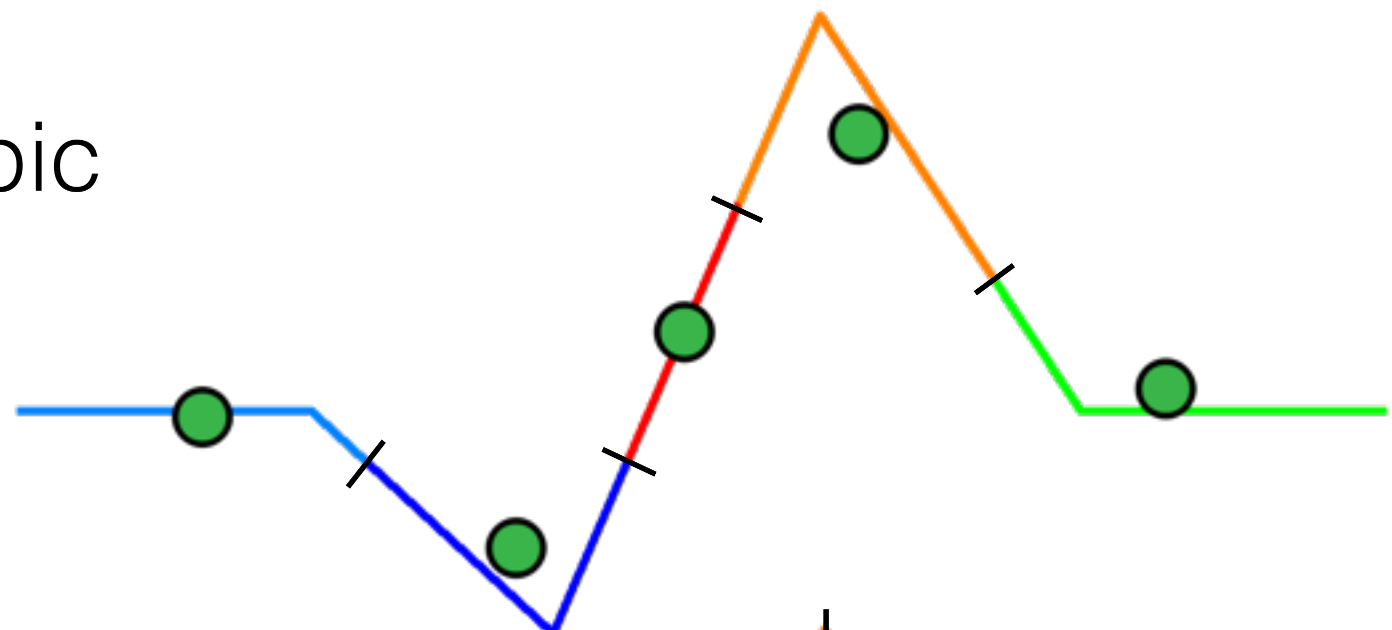
Mahalanobis CVT

1

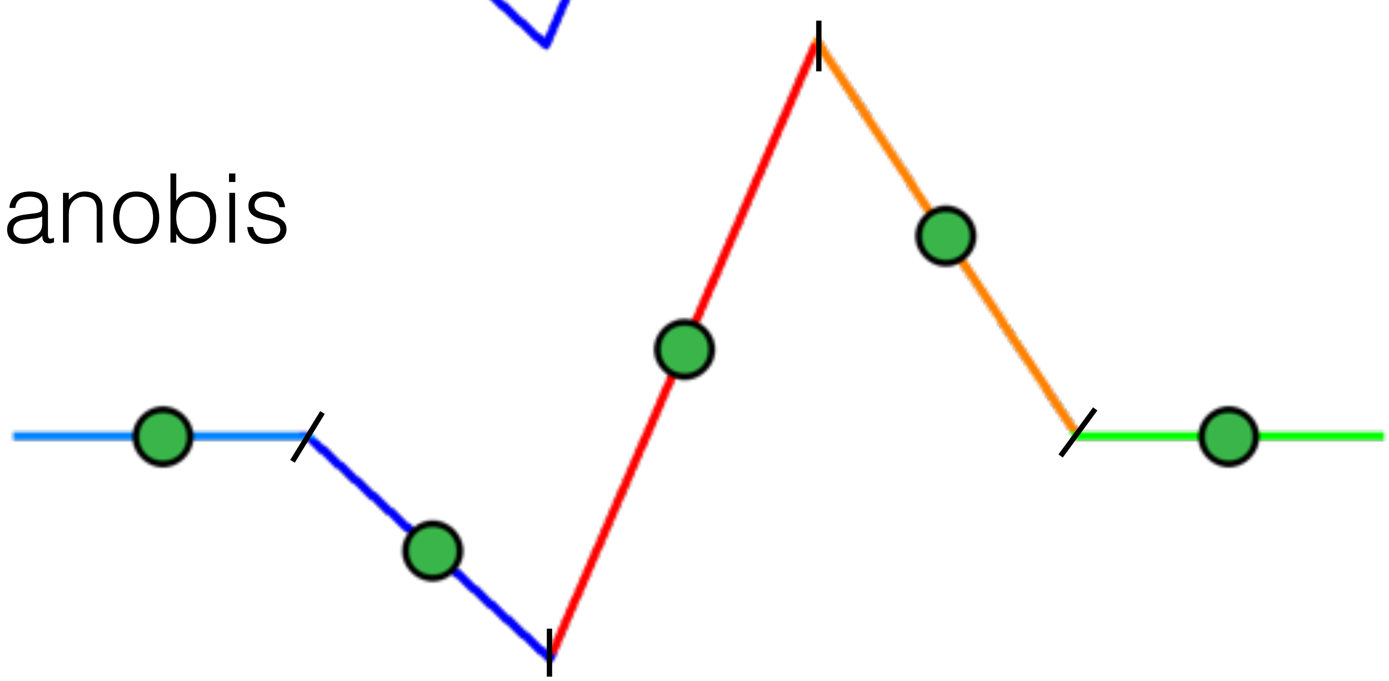


Isotropic vs. Mahalanobis CVT

Isotropic



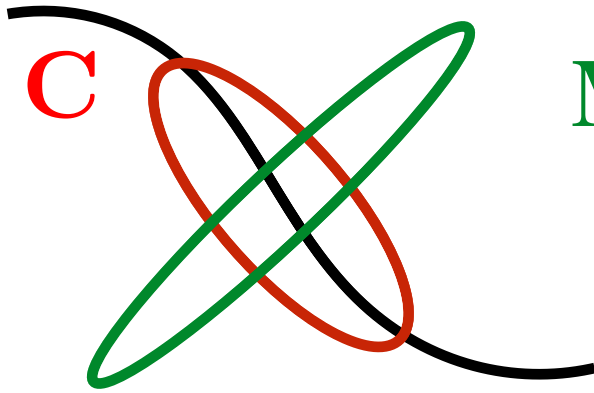
Mahalanobis



Degeneracy

C invertible:

C



$\mathbf{M=|C|^{1/n}\,C^{-1}}$

C singular:

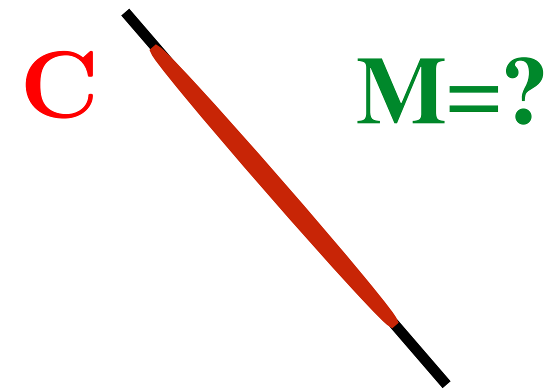
C

M=?

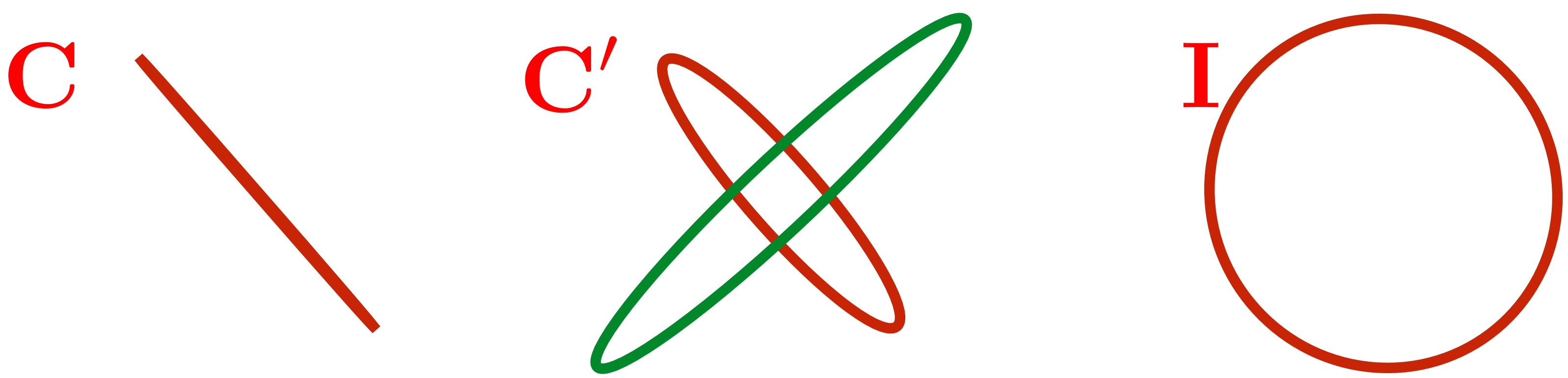
Degeneracy

C invertible:

C singular:

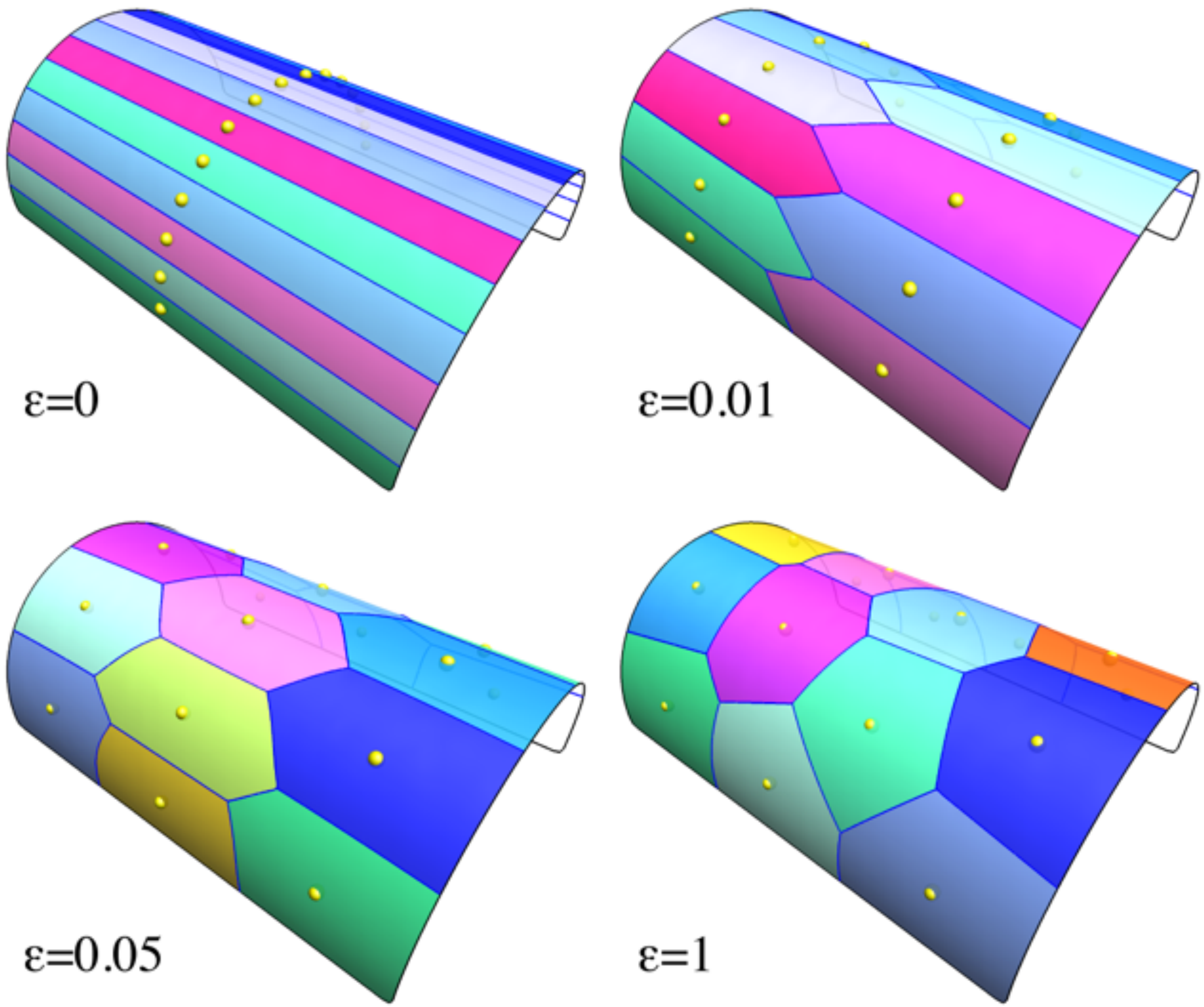


modify C:

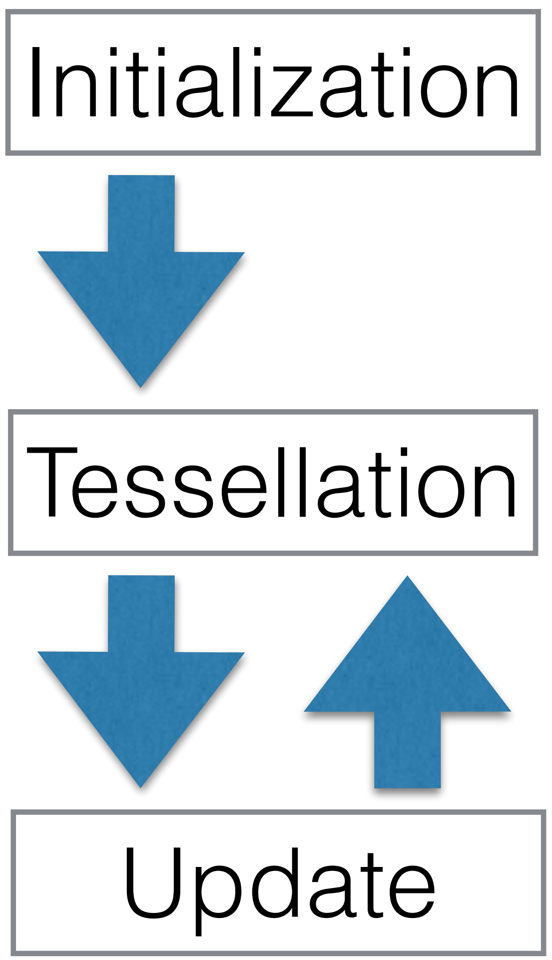


$$\mathbf{C}' = (1-\epsilon)\mathbf{C} + \epsilon\lambda_0\mathbf{I}$$ ($\mathbf{C}$ is psd, symmetric)

Control Anisotropy



Algorithm



$$\mathbf{x}_i = rand(S)$$
$$\mathbf{M}_i = \mathbf{I}$$

FIXED $\mathbf{x}_i, \mathbf{M}_i$

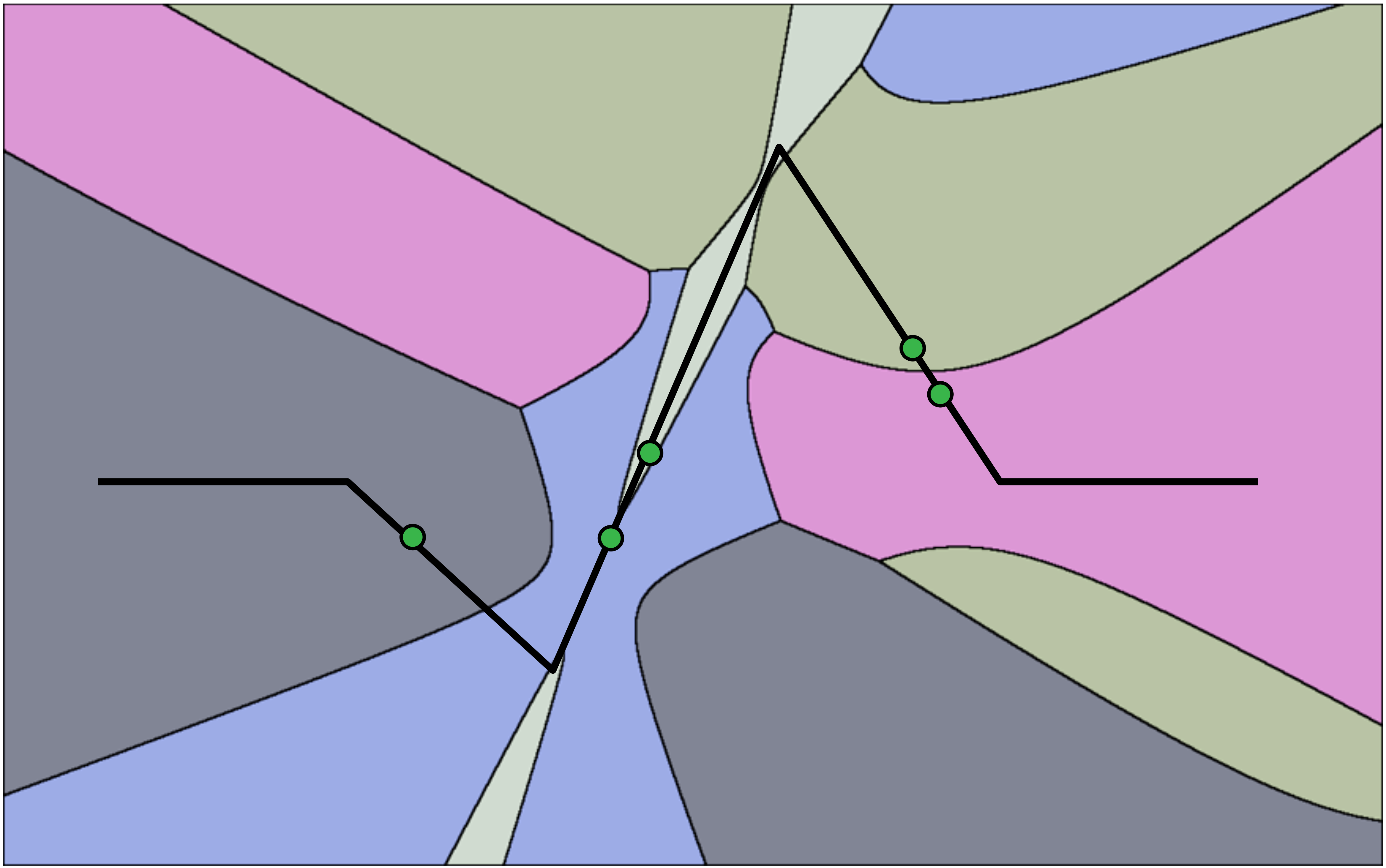
COMPUTE S_i

FIXED S_i

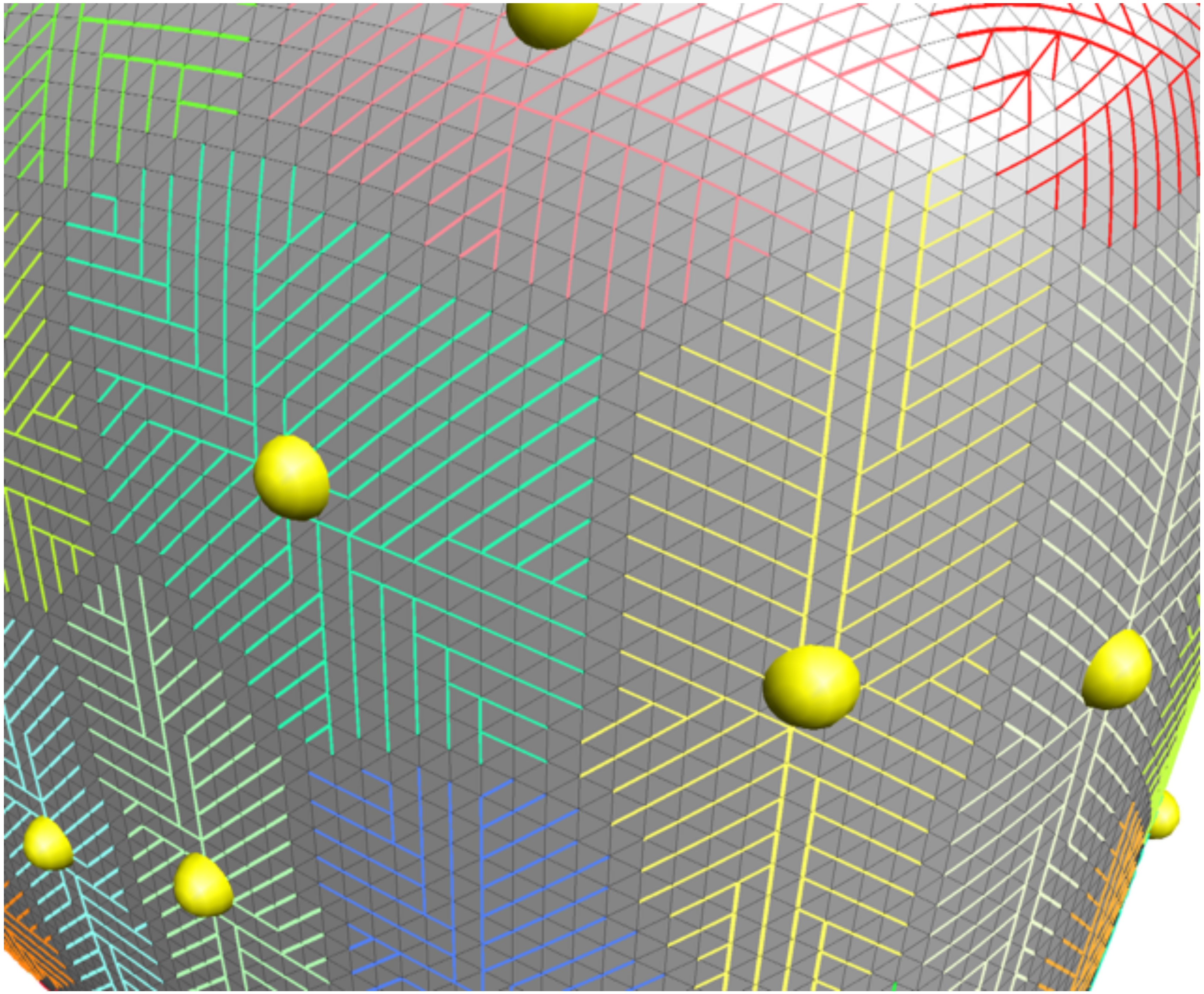
UPDATE $\mathbf{x}_i$

UPDATE $\mathbf{M}_i$

Tessellation

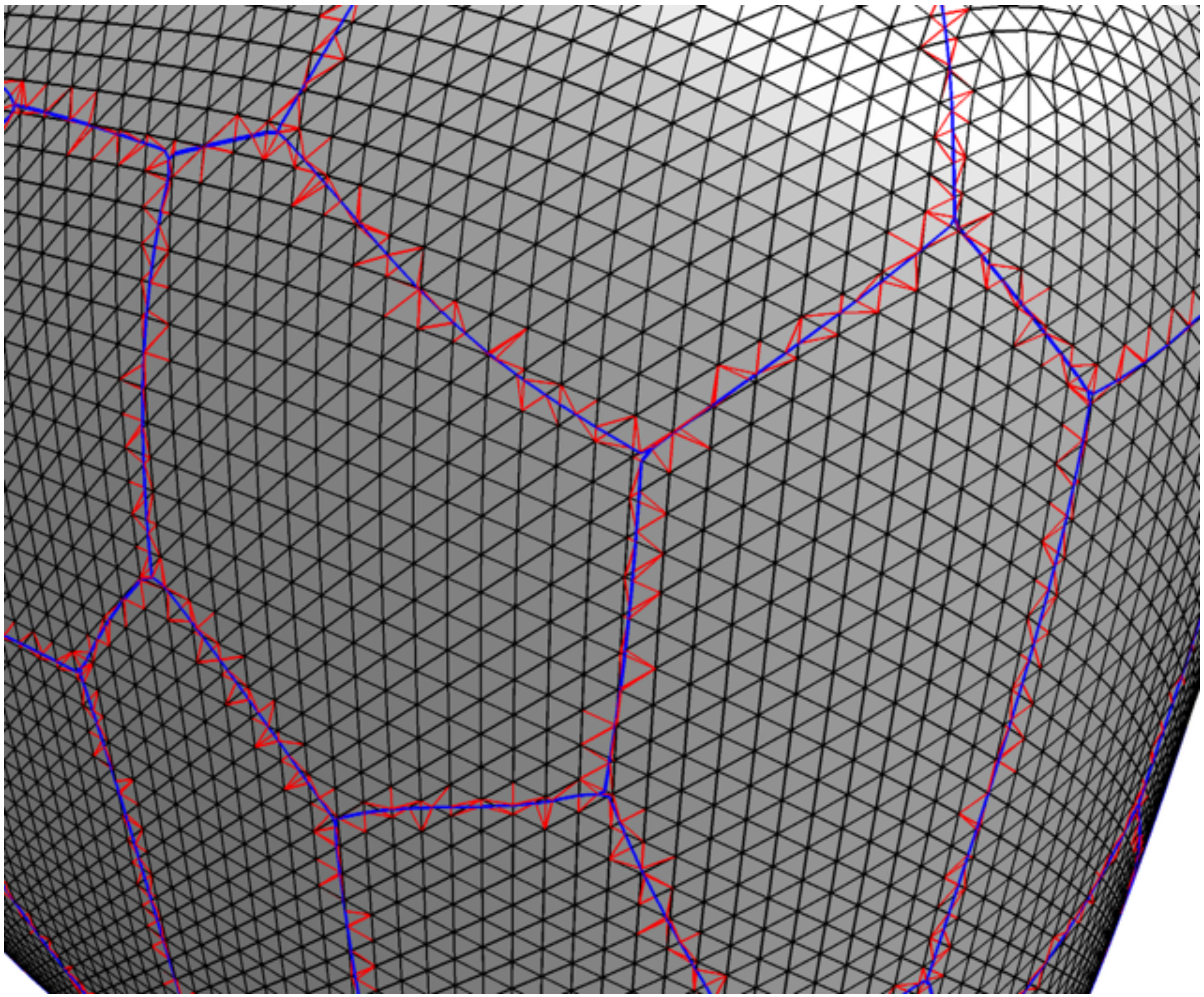


Mesh Tessellation



1. project sites
2. assign vertices

Mesh Tessellation

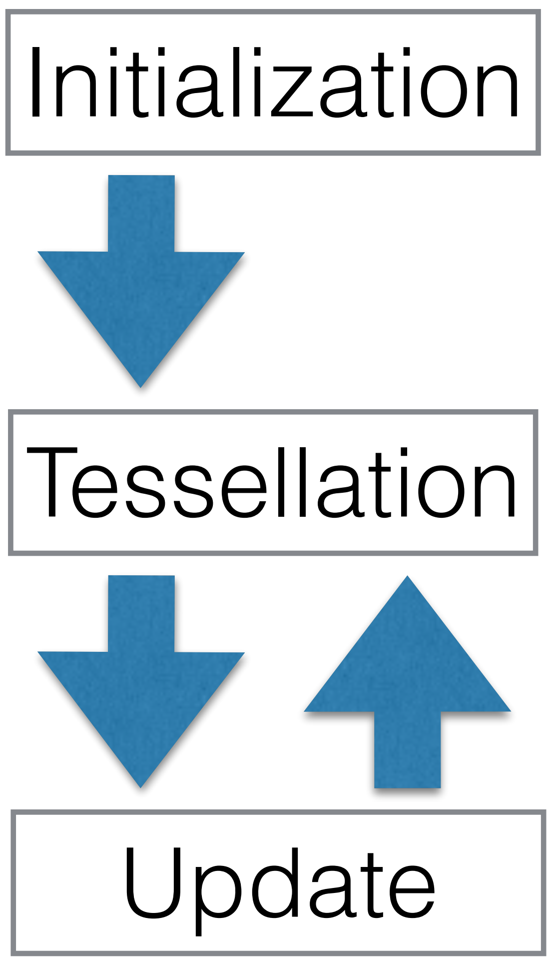


1. bisection on edges
2. subdivide mesh
3. assign faces

Relationship

- Gaussian Mixture Models (GMM) with Expectation Maximization (EM)
- Variational Shape Approximation (VSA) [Cohen-Steiner, Alliez, Desbrun, 2004]

Algorithm / EM

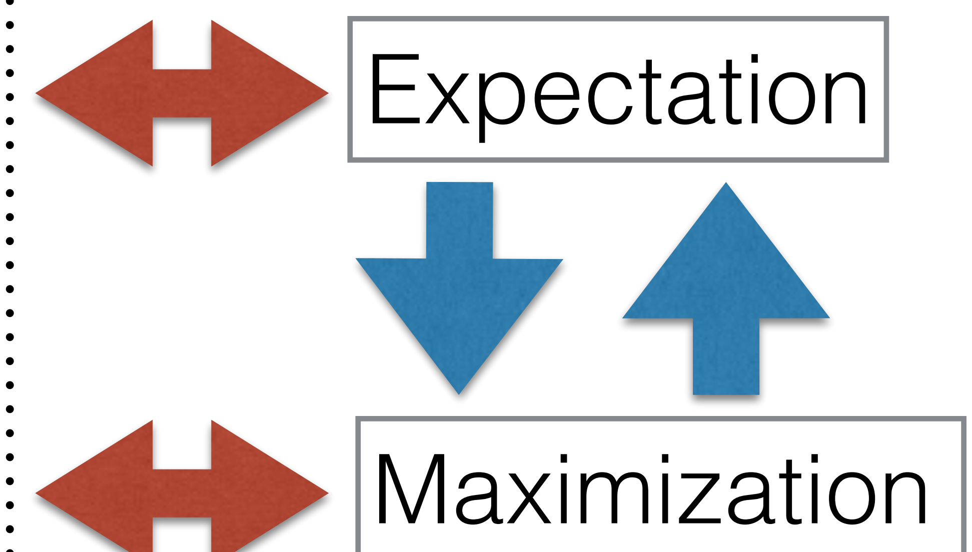


$\mathbf{x}_i = rand(S)$
$\mathbf{M}_i = \mathbf{I}$

FIXED $\mathbf{x}_i, \mathbf{M}_i$
COMPUTE S_i

FIXED S_i
UPDATE $\mathbf{x}_i$
UPDATE $\mathbf{M}_i$

GMM parameters : $\mathbf{x}_i, \mathbf{M}_i$



Algorithm / VSA

Initialization

$\mathbf{x}_i = rand(S)$
$\mathbf{M}_i = \mathbf{I}$

Tessellation

FIXED $\mathbf{x}_i, \mathbf{M}_i$
COMPUTE S_i

Update

FIXED S_i
UPDATE $\mathbf{x}_i$
UPDATE $\mathbf{M}_i$

s.t. *determinant* of M_i is 1

"proxies" :
$\mathbf{x}_i, \mathbf{M}_i$

Flooding

Proxy fitting

s.t. *operator norm* of M_i is 1

Summary

- key messages:
  - introduce metric per cell as **variable**
  - constraining determinant of metric yields **Mahalanobis** distance
  - framework show **connections** to other approaches such as GMM and VSA

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