Diffusion Diagrams
Voronoi Cells and Centroids from Diffusion
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Motivation

Bisectors, Voronoi Cells, Centroids
Fundamental Geometric Primitives
Motivation

Lloyd Iteration

Centroidal Voronoi Tessellation
Motivation
Geometric Computation on Curved Surfaces
Distances: Geodesics

- Voronoi cells and centroids for curved surfaces are mostly based on geodesics

- Options:
  - fast marching
  - exact polyhedral distance
  - geodesics in heat
Distances: Geodesics

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  linear approx. of heat + normalization of grad. + integration
Distance from Heat Diffusion

- Same for Euclidean domains
- Accurate: Cells converge quadratically under mesh refinement for Euclidean and Spherical domains
- Stable: Centroids are better behaved
- Fast: Diffusion is easy to compute for triangle meshes and can be accelerated by localization and parallelization
Distance from Heat Diffusion

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Heat Diffusion

Distribution of heat starting from a given heat distribution after time $t$. 

$p$
Heat Diffusion

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Distribution of heat starting from a given heat distribution after time $t$. 

$k(t, p, q)$
Heat Diffusion

Distribution of heat starting from a given heat distribution after time $t$
Discrete Heat Diffusion

Only need Laplace-Beltrame Operator
Discrete Heat Diffusion

Only need Laplace-Beltrame Operator for Semi-Implicit Time Stepping

$$(I - tM^{-1}L_C)u = h$$
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Discrete Heat Diffusion

Piecewise Linear Initial Conditions
Discrete Heat Diffusion

Piecewise Linear Heat Distribution

\[(I - tM^{-1}L_C)u = h\]
\[(M - tL_C)u = Mh\]

**Discrete Heat Diffusion**

Sparse Symmetric Linear System
Precompute Cholesky Factors
Voronoi Cells from Heat
Voronoi Cells

- Given Points
Voronoi Cells

- Given Points
- Bisectors defines Voronoi Cell
Heat Cells

• Given Points
Heat Cells

- Given Points
- Compute heat diffusion from each of the points
Heat Cells

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Heat Cells

- Given Points
- Compute heat diffusion from each of the points
- Define cell based on maximum heat
Heat Cells

- Given Points
- Compute heat diffusion from each of the points
- Define cell based on maximum heat
- Maximum heat = minimum distance for radial heat kernels
Accuracy / Convergence
Heat Cells - Numerics

• Consider perfect (smooth) sphere with two generators

• Bisectors are known analytically
Heat Cells - Numerics

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Heat Cells - Numerics

• Consider perfect (smooth) sphere with two generators
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• Consider triangulated spheres converging to smooth sphere
Heat Cells - Numerics

- Consider perfect (smooth) sphere with two generators
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- Consider triangulated spheres converging to smooth sphere
  - Irregular triangles
  - Similar size
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Heat Cells vs. Geodesic Voronoi Cells

• Compared to Polyhedral Distance
  • Same asymptotic accuracy (relative to smooth surface)
  • Much faster

• Compared to Geodesics in Heat
  • Better accuracy (quadratic vs. linear)
  • Faster (only one of two linear solves)
Centroids from Heat
Centroids

= Center of Mass
Riemannian Center of Mass

$0 = \int_C (\mathbf{x} - \mathbf{c}) \, d\mathbf{x}$

$c = \arg \min_{\mathbf{y}} \int_C (\mathbf{x} - \mathbf{y})^T (\mathbf{x} - \mathbf{y}) \, d\mathbf{x}$

Riemannian Center of Mass

= Minimizer of Sum of Squared Distances
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\[ 0 = \int_{\mathcal{C}} (\mathbf{x} - \mathbf{c}) \, d\mathbf{x} \]

\[ \mathbf{c} = \arg \min_{\mathbf{y}} \int_{\mathcal{C}} \|\mathbf{x} - \mathbf{y}\|^2 \, d\mathbf{x} \]
Riemannian Center of Mass

= Minimizer of Sum of Squared Distances

\[ 0 = \int_C (\mathbf{x} - \mathbf{c}) \, dx \]

\[ \mathbf{c} = \arg\min_{\mathbf{y}} \int_C d^2(\mathbf{x}, \mathbf{y}) \, dx \]

Riemannian Center of Mass

= Minimizer of Sum of Squared Distances
Heat Centroids
Heat Centroids

\[ c = \arg \min_y \int_c d^2(x, y) \, dx \]

\[ c(t) = \arg \max_y \int_c k(t, x, y) \, dx \]

Heat Centroids

= Maximizer of Heat in Cell
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Heat Centroids

= Maximum Heat Remaining Cell
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Heat Centroids

= Maximum Heat Remaining Cell
Similarity to Euclidean Centroids
\[ c = \arg \min_y \int_C \|x - y\|^2 \, dx \]

\[ c(t) = \arg \max_y \int_C k(t, x, y) \, dx \]

**Heat Centroids**

Euclidean domain
\[ c = \arg \min_y \int_C \| x - y \|^2 \, dx \]

\[ c(t) = \arg \max_y \int_C k(t, x, y) \, dx \]

\[ k(t, x, y) = \frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{\|x - y\|^2}{4t} \right) \, dx \]

Heat Centroids

Euclidean domain
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Heat Centroids

Euclidean domain
Taylor expansion
\[ c = \arg \min_y \int_C \|x - y\|^2 \, dx \]

\[ c(t) = \arg \max_y \int_C \exp \left( -\frac{\|x - y\|^2}{4t} \right) \, dx \]

\[ \exp \left( -\frac{\|x - y\|^2}{4t} \right) \, dx = 1 - \frac{\|x - y\|^2}{4t} + \frac{\|x - y\|^4}{32t^2} - \ldots \]

**Heat Centroids**

Euclidean domain

Taylor expansion
Heat Centroids

Euclidean domain

\[ c = \arg\min_y \int_C \|x - y\|^2 \, dx \]

\[ c(t) = \arg\max_y \int_C 1 - \frac{\|x - y\|^2}{4t} + \frac{\|x - y\|^4}{32t^2} - \ldots \, dx \]
\[ c = \arg \min_y \int_C \|x - y\|^2 \, dx \]

\[ c(t) = \arg \max_y \int_C 1 \times \frac{\|x - y\|^2}{4t} + \frac{\|x - y\|^4}{32t^2} - \ldots \, dx \]

Heat Centroids
Euclidean domain
\[ c = \arg \min_y \int_C \| x - y \|^2 \, dx \]

\[ c(t) = \arg \min_y \int_C \frac{\| x - y \|^2}{4t} - \frac{\| x - y \|^4}{32t^2} + \ldots \, dx \]

Heat Centroids
Euclidean domain
\[ c = \arg \min_y \int_C \|x - y\|^2 \, dx \]

\[ c(t) = \arg \min_y \int_C \frac{\|x - y\|^2}{4t} - \frac{\|x - y\|^4}{32t^2} + \ldots \, dx \]
\[ c = \arg \min_y \int_C \|x - y\|^2 \, dx \]

\[ c(t) = \arg \min_y \int_C \|x - y\|^2 - \frac{\|x - y\|^4}{8t} + \ldots \, dx \]

**Heat Centroids**

Euclidean domain
Stability of Heat Centroids
Stability of Centroids - Cells
Stability of Geodesic Centroids
Stability of Geodesic Centroids
Stability of Heat Centroids
Computational Speed
Linear Solve vs. Polyhedral Geodesics

\[ CC^T u = Mh \]

Sparse Cholesky

\[ CC^T = (M - tL_C) \]

\( t \text{ [ms]} \)

#vertices

Polyhedral geodesics
Diffusion method
Linear Solve vs. Polyhedral Geodesics

- Polyhedral geodesics
- Diffusion method

$t$ [ms]

\[
CC^T u = Mh
\]

\[
CC^T = (M - tL_C)
\]

Without parallelization

Sparse Cholesky

#vertices
Localized Sparse Solve

Intermediate solution for Sparse RHS is sparse!
Localized Sparse Solve

Eventual Solution is not sparse!

\[ Cx = y \]

\[ \begin{pmatrix}
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \end{pmatrix} \quad x = \quad \begin{pmatrix}
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \bullet \\
  \end{pmatrix} \]
Localized Sparse Solve

$C x = y$

$\begin{pmatrix}
\bullet \\
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\end{pmatrix}
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\end{pmatrix}$

... but can be localized to subset
Localized Sparse Solve - Error
Results
Centroidal Heat Tessellation (CHT)
Centroidal Heat Tesselations (CHT)

• Applicable to large meshes

• Intrinsic, so no problem with self intersection

• Equalizes area
Anisotropic CHT

Lift points + normals to 6D
Only Laplace-Beltrami operator matrix changes
CHT for Meshing Point Clouds

Applies to every discrete representation with Laplace operator
Natural neighbor coordinates

• Define barycentric coordinates based on area of Voronoi cells
Natural neighbor coordinates
Natural neighbor coordinates
Recall: Distance from Heat Diffusion

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Thanks!

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  • Philipp Herholz
  • Felix Haase
  • Boris Springborn

• Funding agencies
  • ERC (“XShape”)
  • Einstein Foundation
Heat Centroids - Numerics

- Irregular triangulation of disk
Heat Centroids - Numerics

- Irregular triangulation of disk
- Consider nice polygons
Heat Centroids - Numerics

- Irregular triangulation of disk
- Consider not so nice polygons
Heat Diagram
Assign triangles based on heat in vertex
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