Localized solutions of sparse linear systems for geometry processing

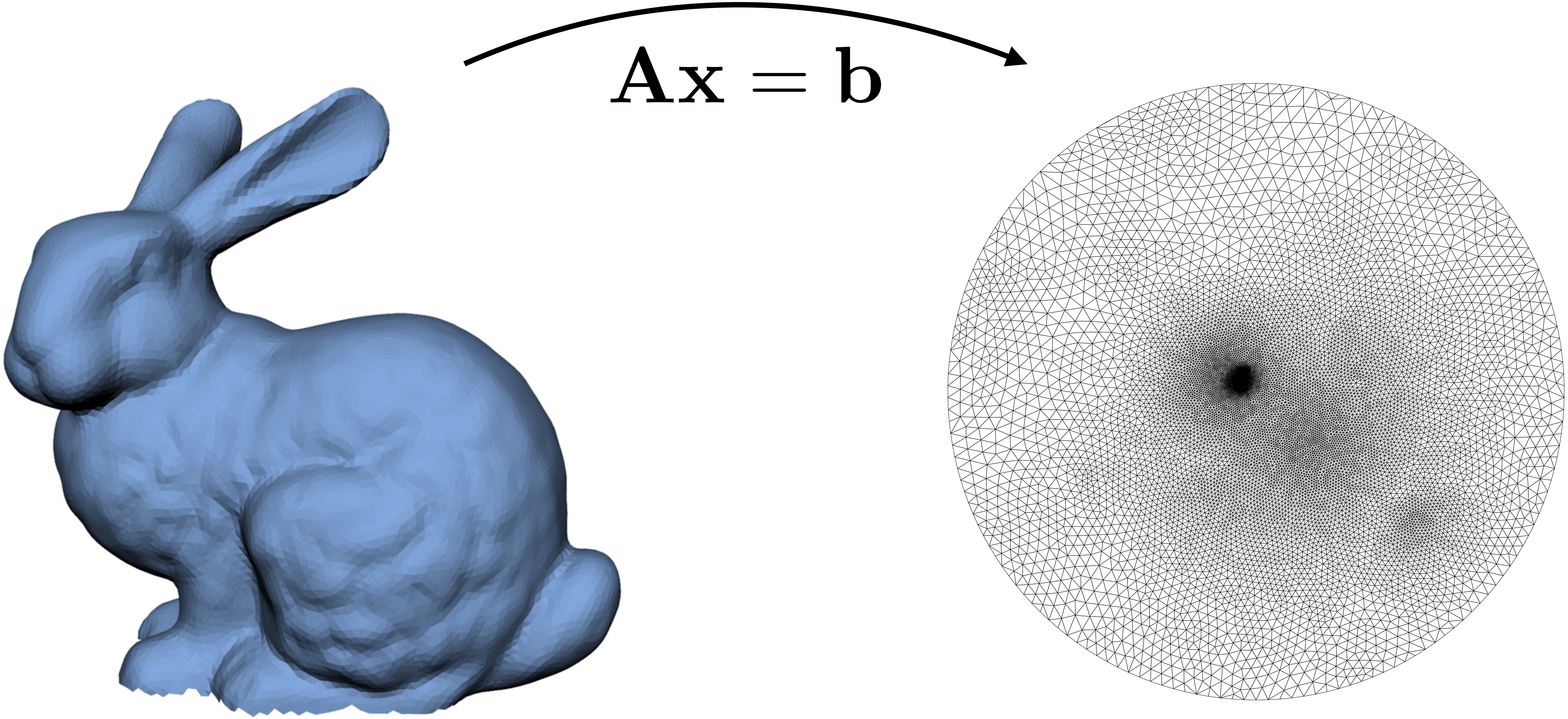
Philipp Herholz, Timothy A. Davis, Marc Alexa

Linear solvers in geometry processing

- Many applications require repeated solutions of linear systems.

Discrete conformal parameterization

$$\mathbf{Ax} = \mathbf{b}$$

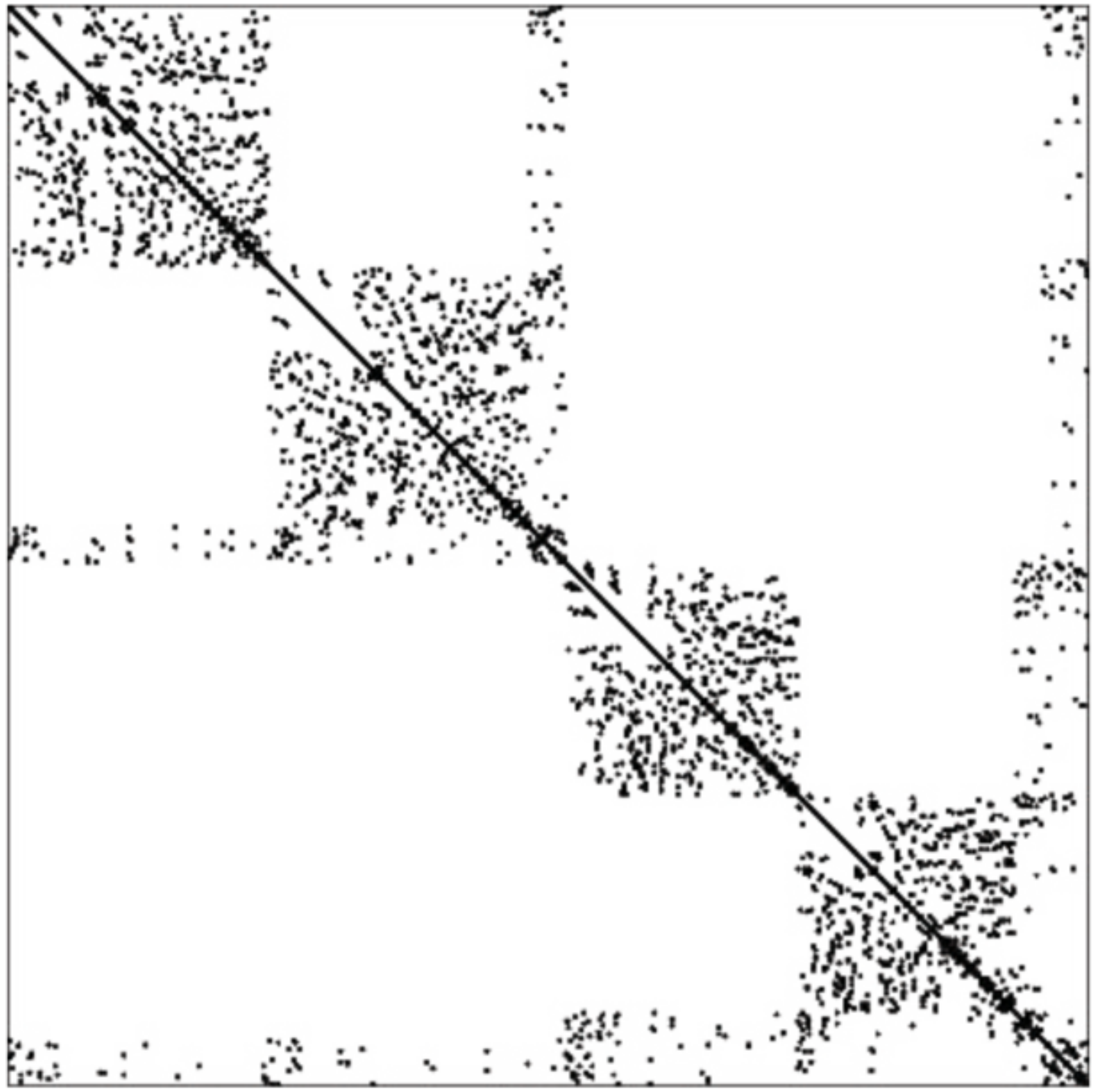


Discrete conformal parameterization

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$\mathbf{A}$ is typically:

- sparse
- symmetric
- positive semi-definite



Cholesky factorization

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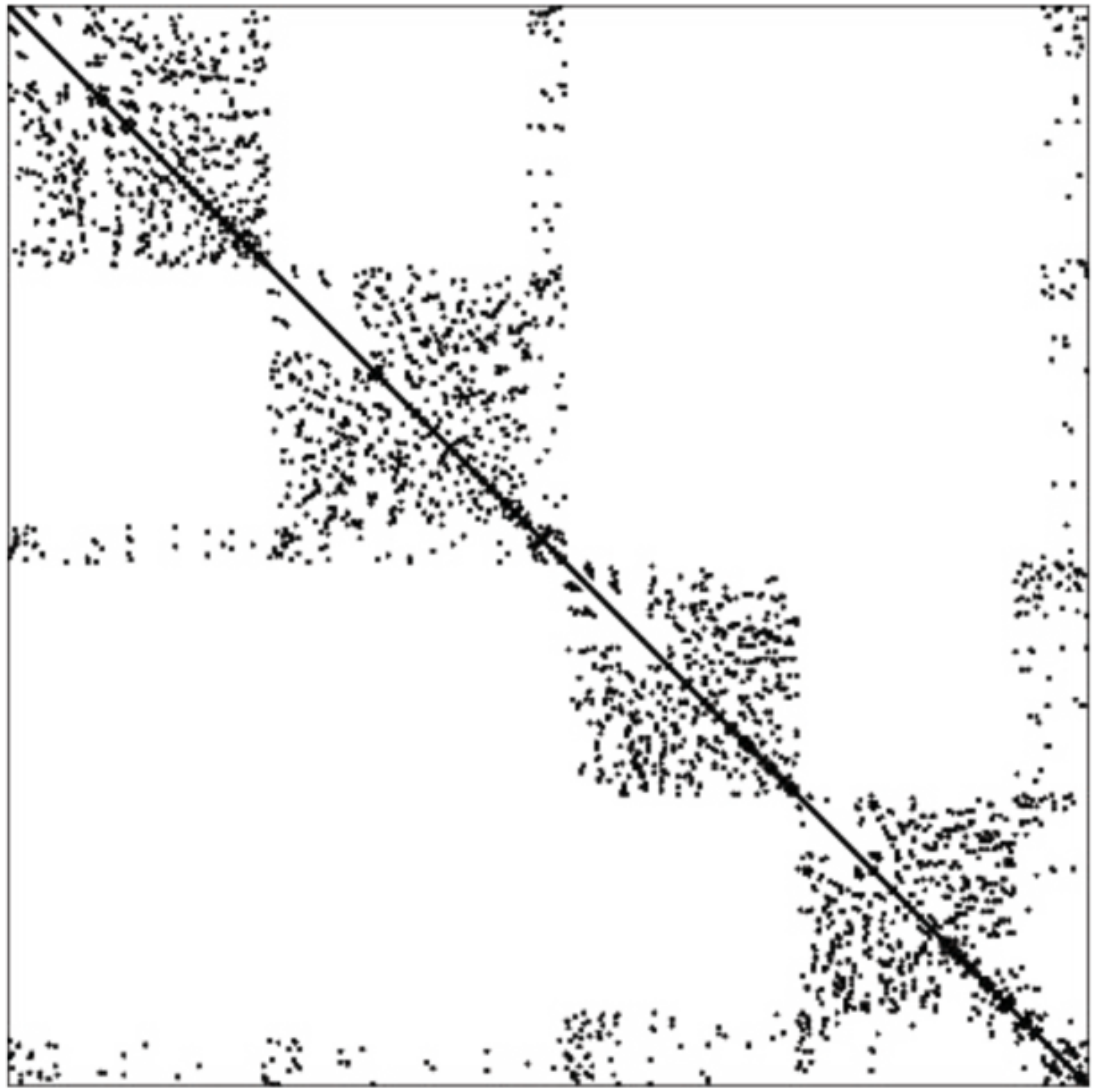


Sparse Cholesky factorization can be applied!

Cholesky factorization

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A} = \mathbf{L} \times \mathbf{D} \times \mathbf{L}^T$$

Cholesky factorization

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{L}\mathbf{D}\mathbf{L}^T\mathbf{x} = \mathbf{b}$$

Cholesky factorization

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{LD}\underbrace{\mathbf{L}^T\mathbf{x}}_{\mathbf{y}} = \mathbf{b}$$

Cholesky factorization

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{L}\mathbf{D}\underbrace{\mathbf{L}^T\mathbf{x}}_{\mathbf{y}} = \mathbf{b}$$

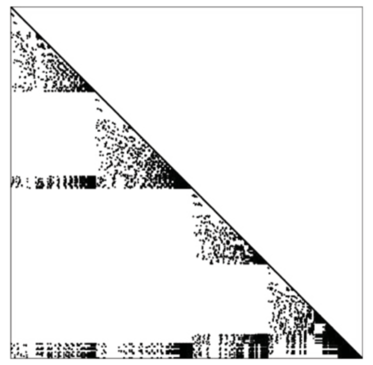
$$\mathbf{L}\mathbf{D}\mathbf{y} = \mathbf{b} \; (1)\text{Forward solve}$$

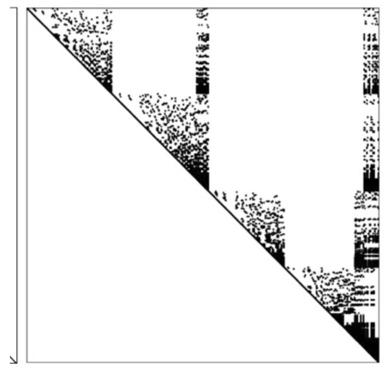
$$\mathbf{L}^T\mathbf{x} = \mathbf{y} \; (2)\text{Back solve}$$

Cholesky factorization

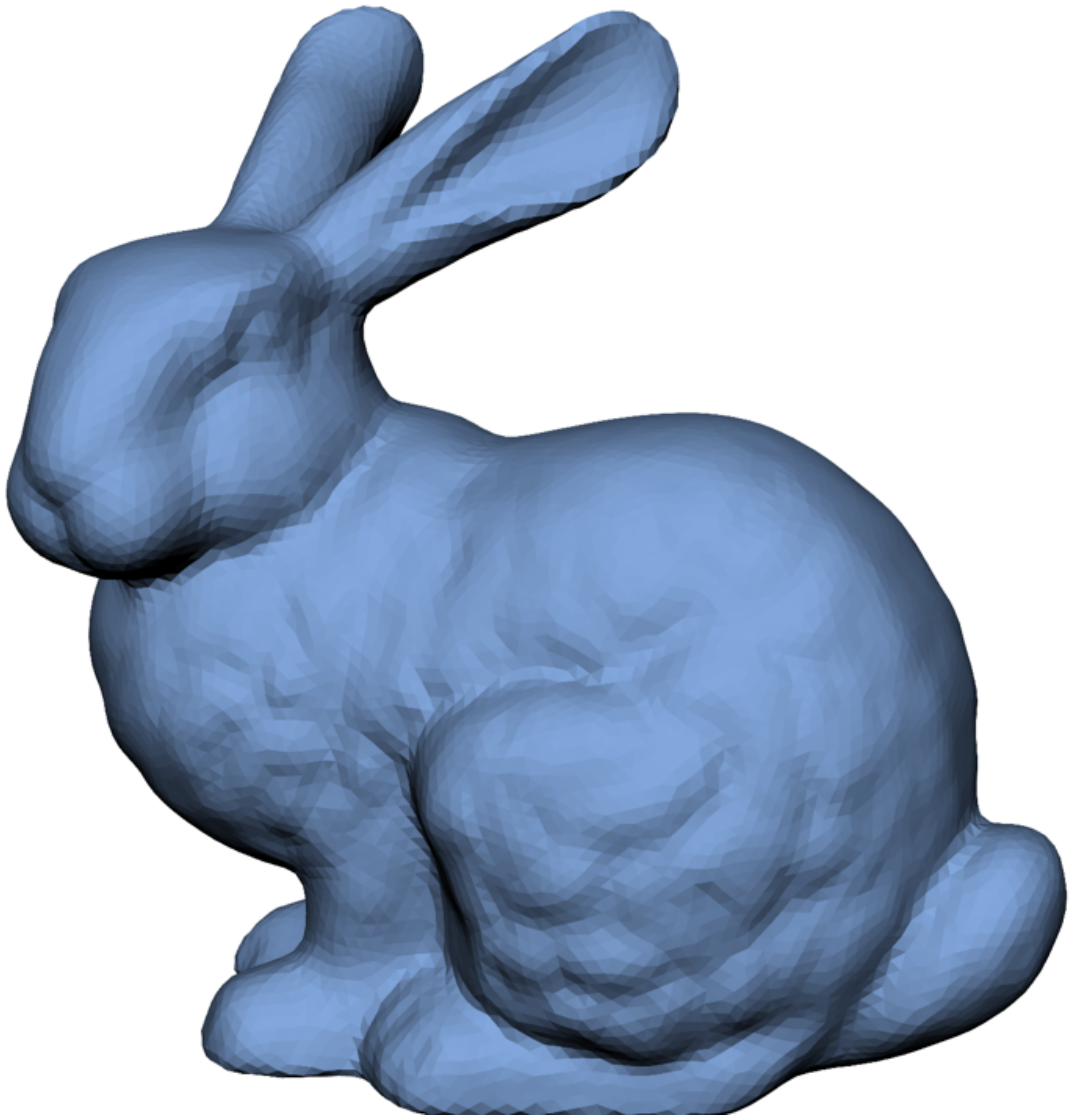
$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{LD}\underbrace{\mathbf{L}^T\mathbf{x}}_{\mathbf{y}} = \mathbf{b}$$

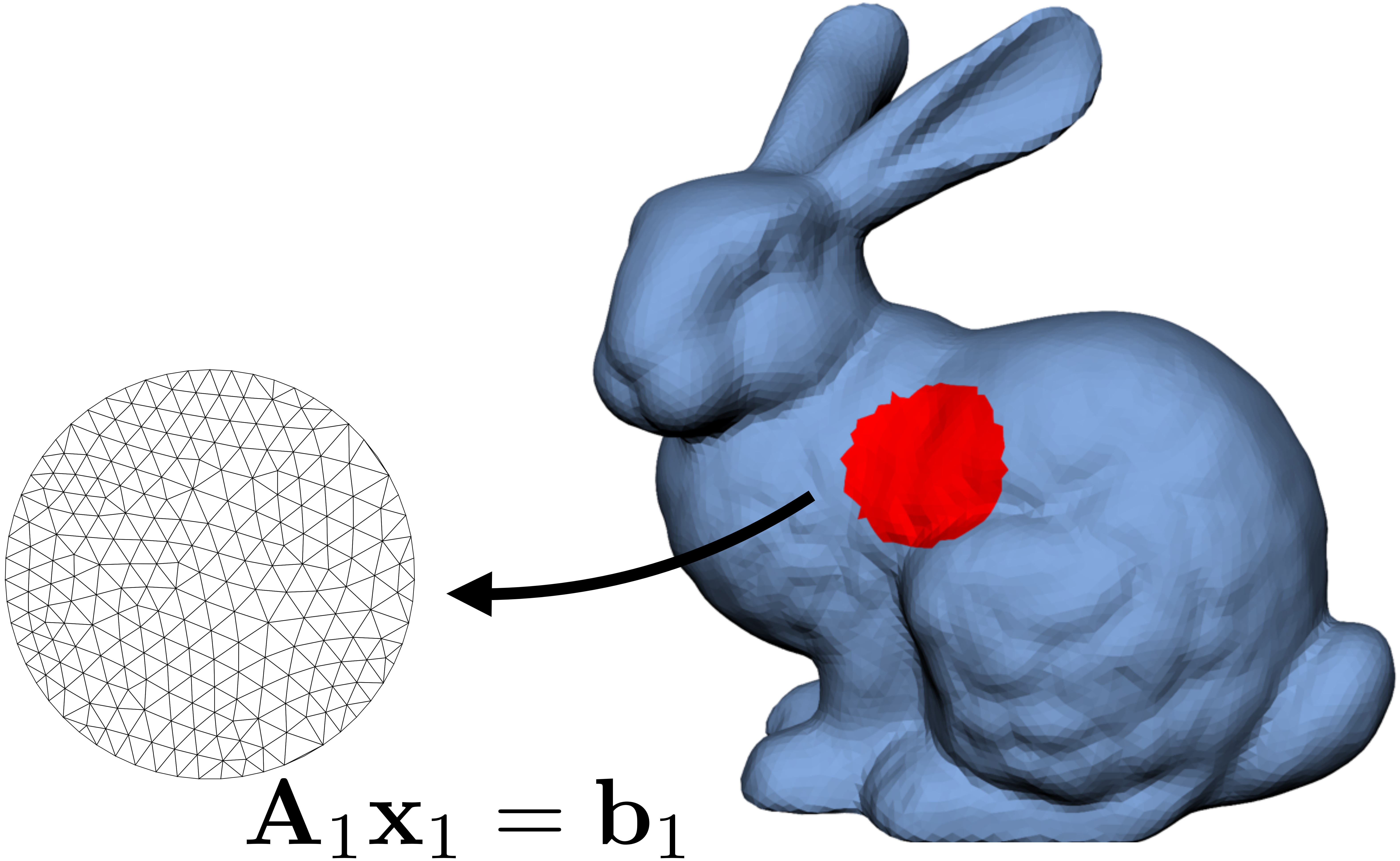
 $\mathbf{y} = \mathbf{b}$ (1)Forward solve

 $\mathbf{x} = \mathbf{y}$ (2)Back solve

Local discrete conformal parameterization



Local discrete conformal parameterization



$$\mathbf{A}_1 \mathbf{x}_1 = \mathbf{b}_1$$

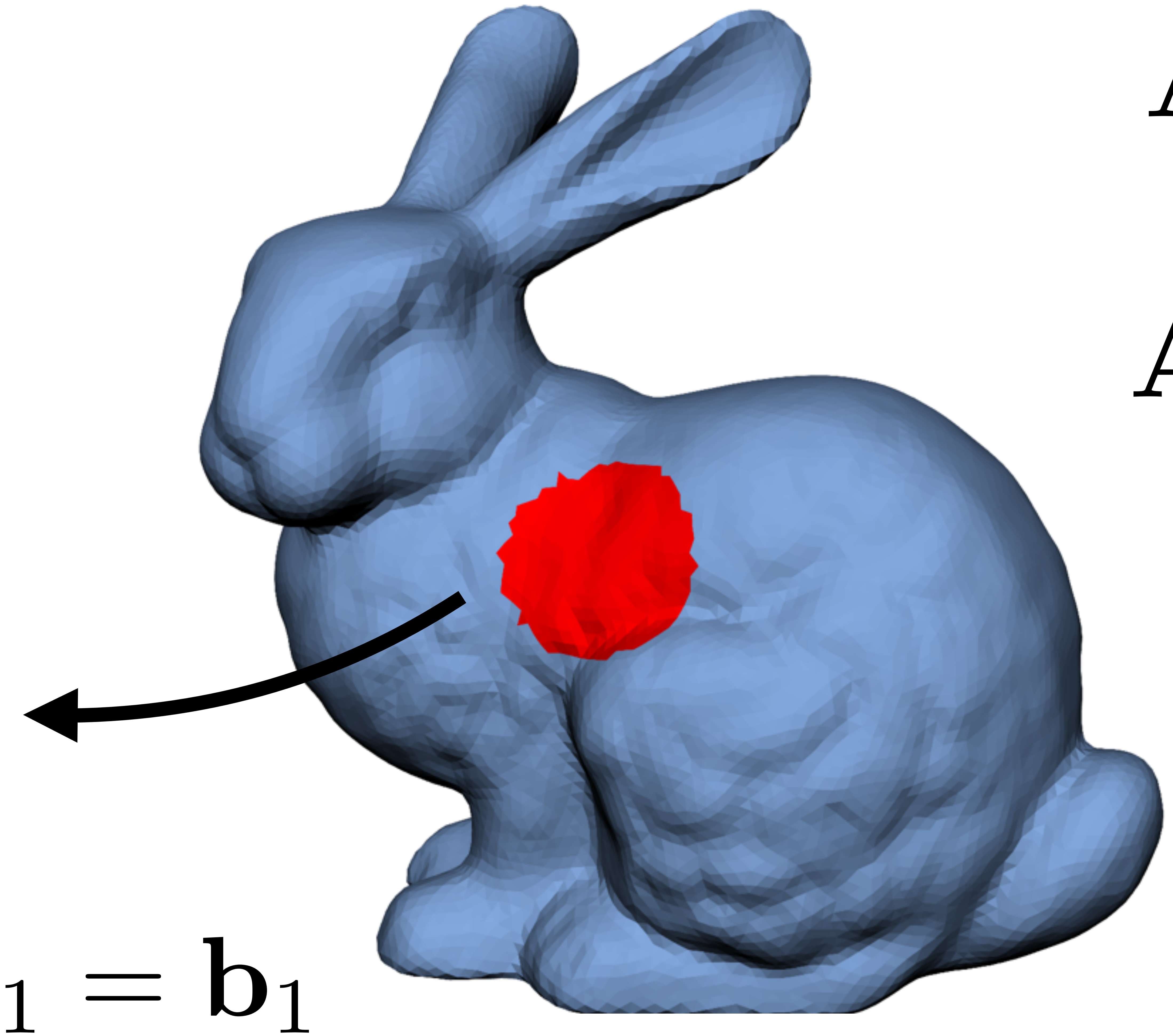
Local discrete conformal parameterization

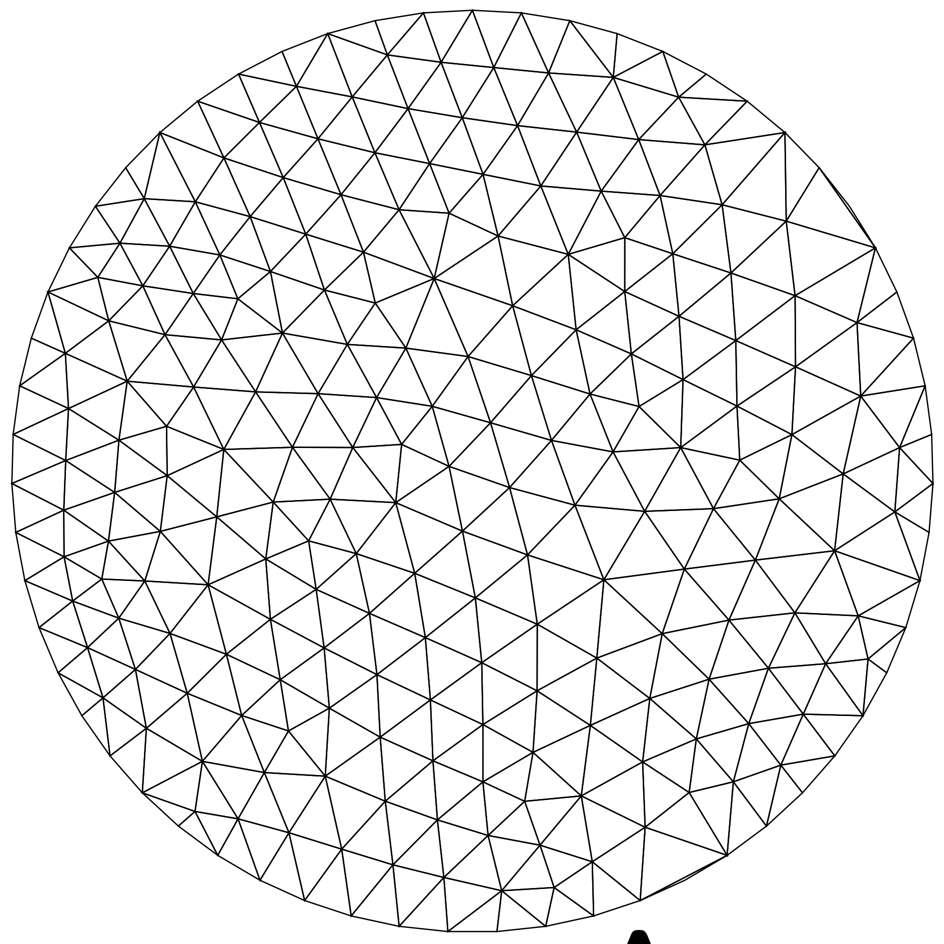
$$\mathbf{A}_1 = \mathbf{A}\Big($$  $$\Big)$$

$$\mathbf{A}_1 = \mathbf{LDL}^T$$

$$\mathbf{LDy} = \mathbf{b}_1$$

$$\mathbf{L}^T\mathbf{x}_1 = \mathbf{y}$$





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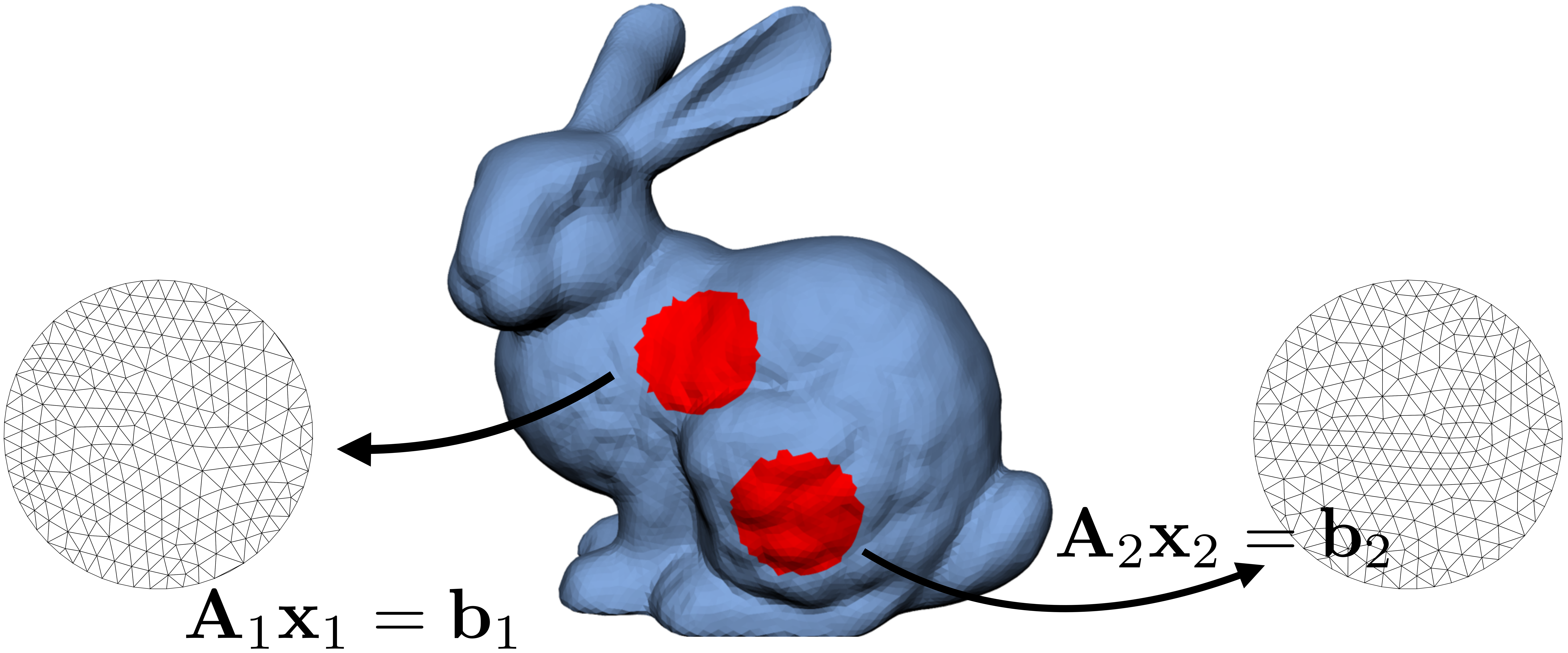
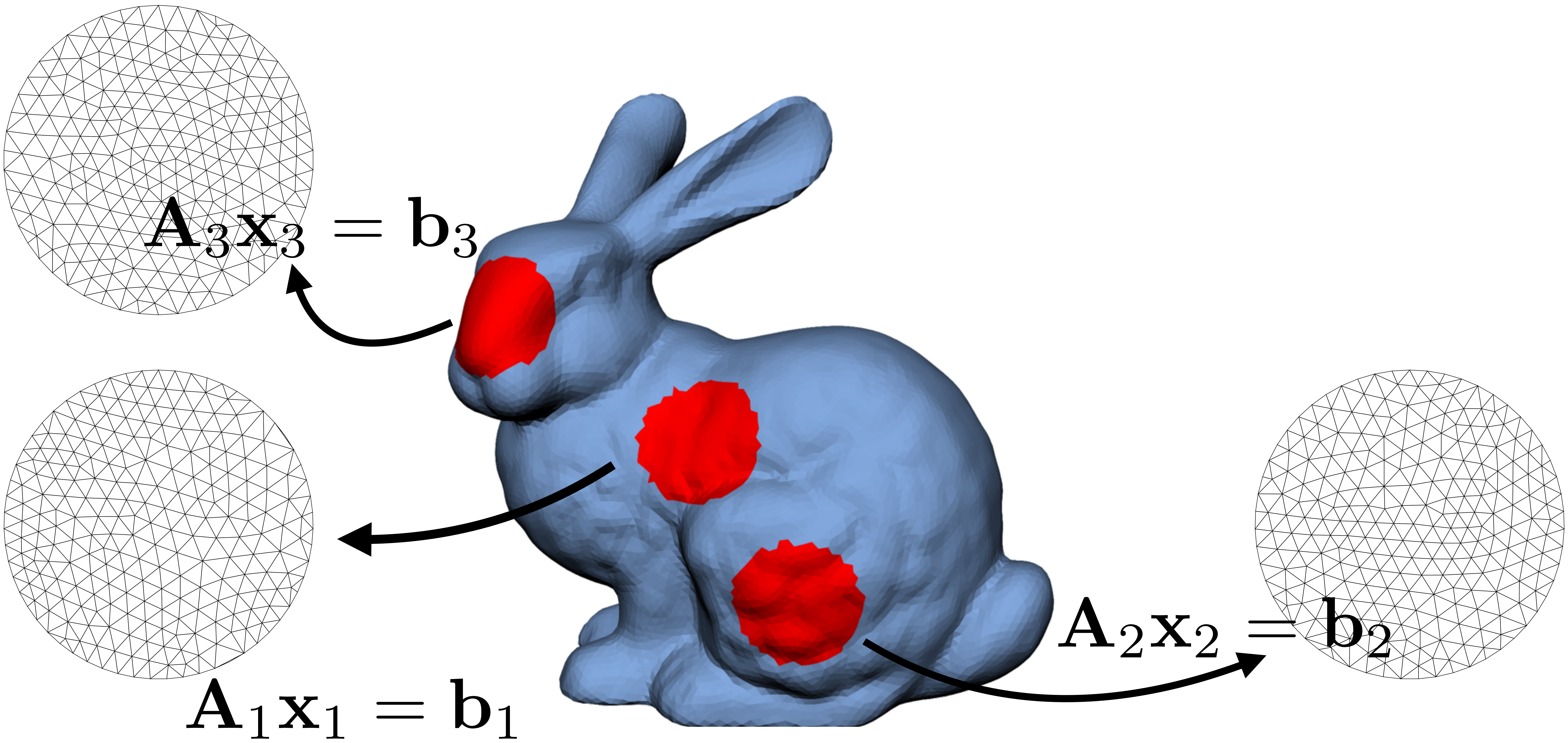
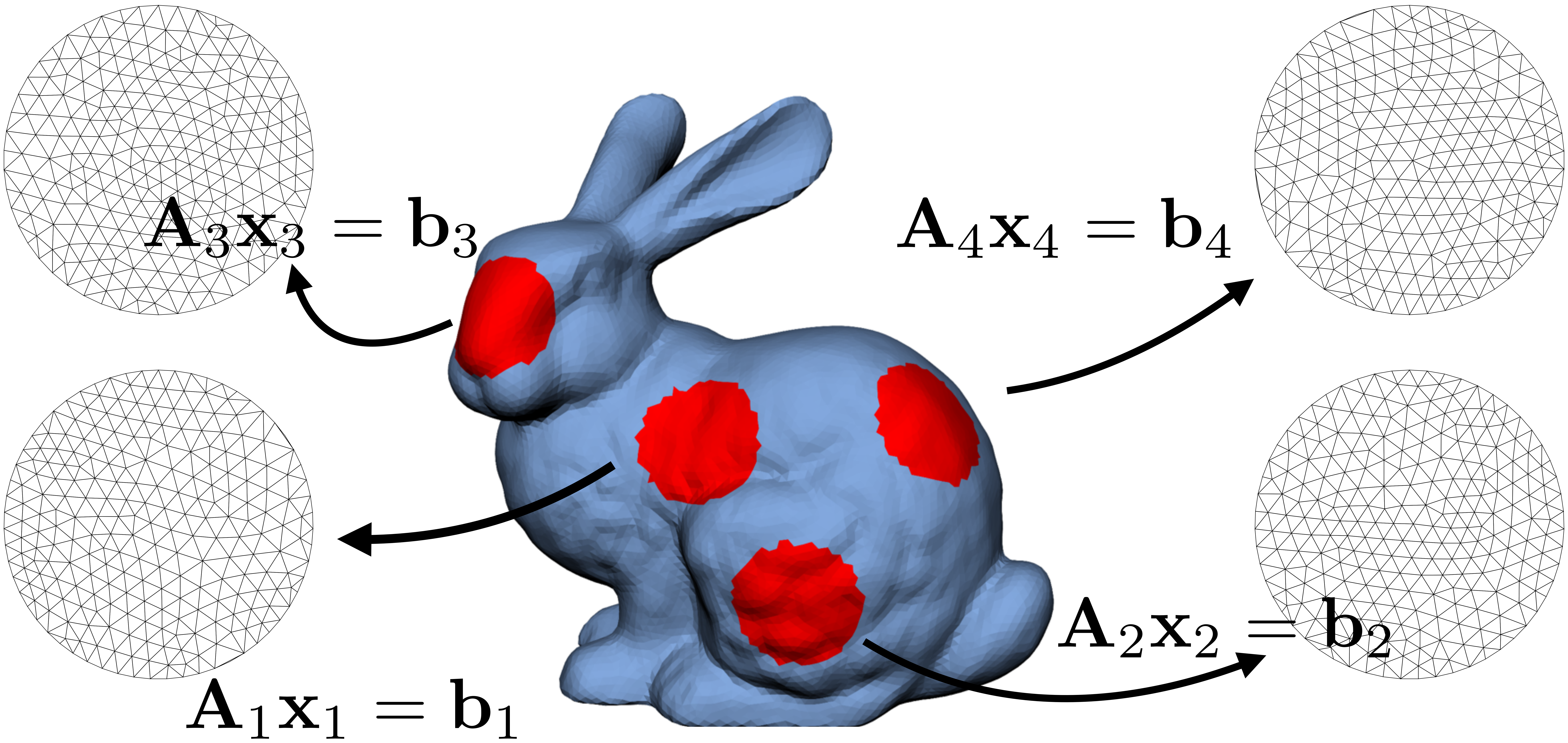
Local discrete conformal parameterization



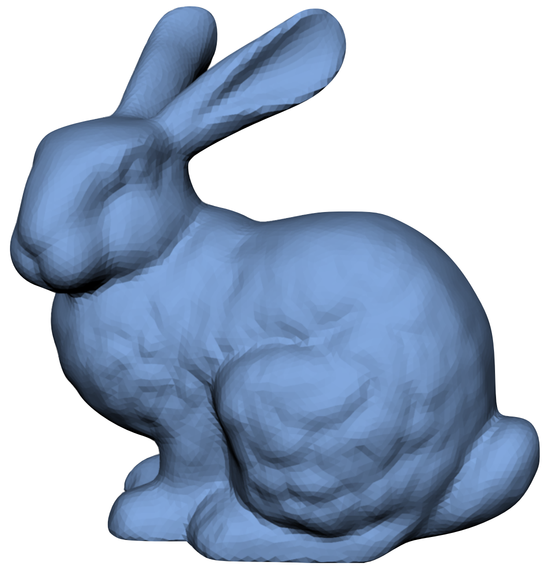
Local discrete conformal parameterization

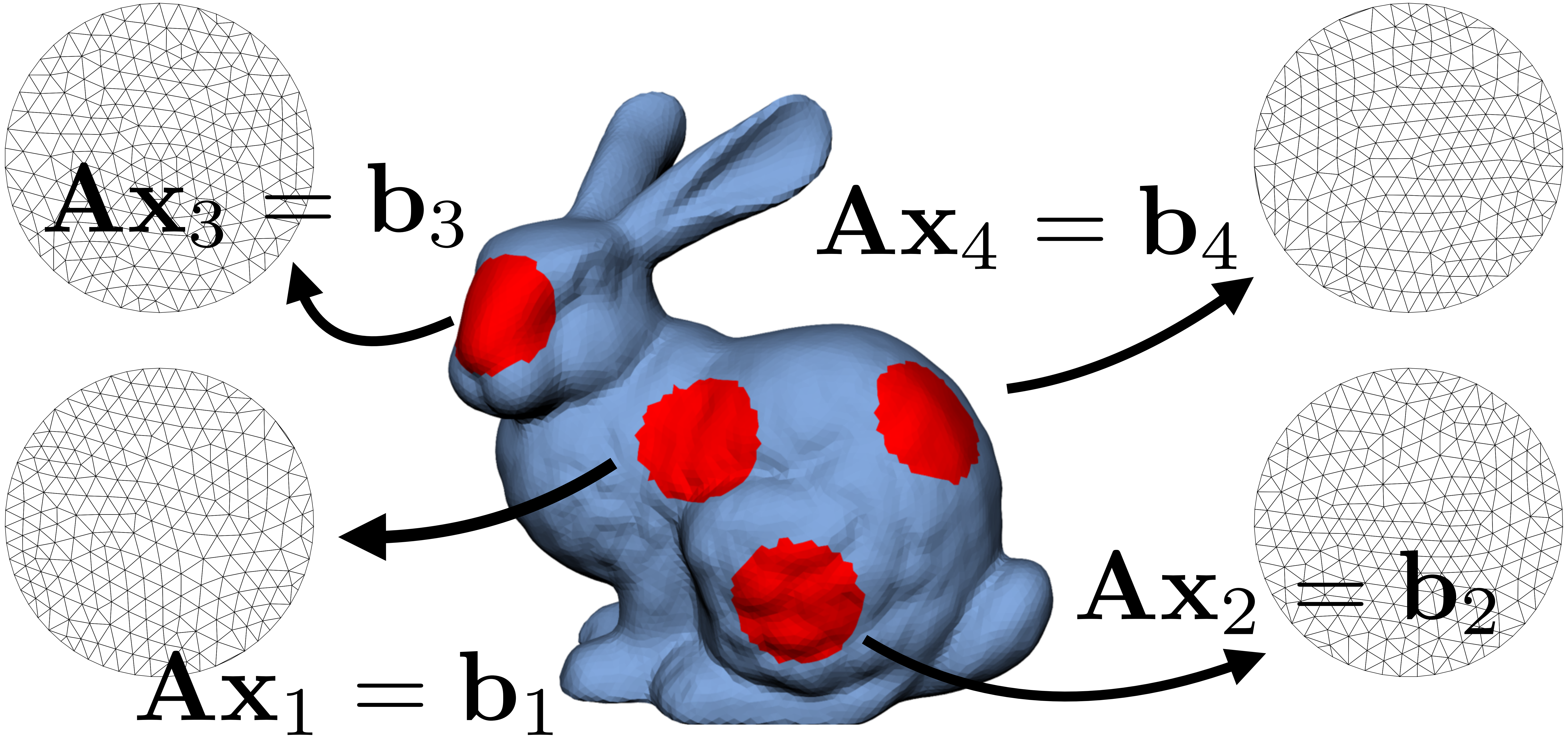


Local discrete conformal parameterization



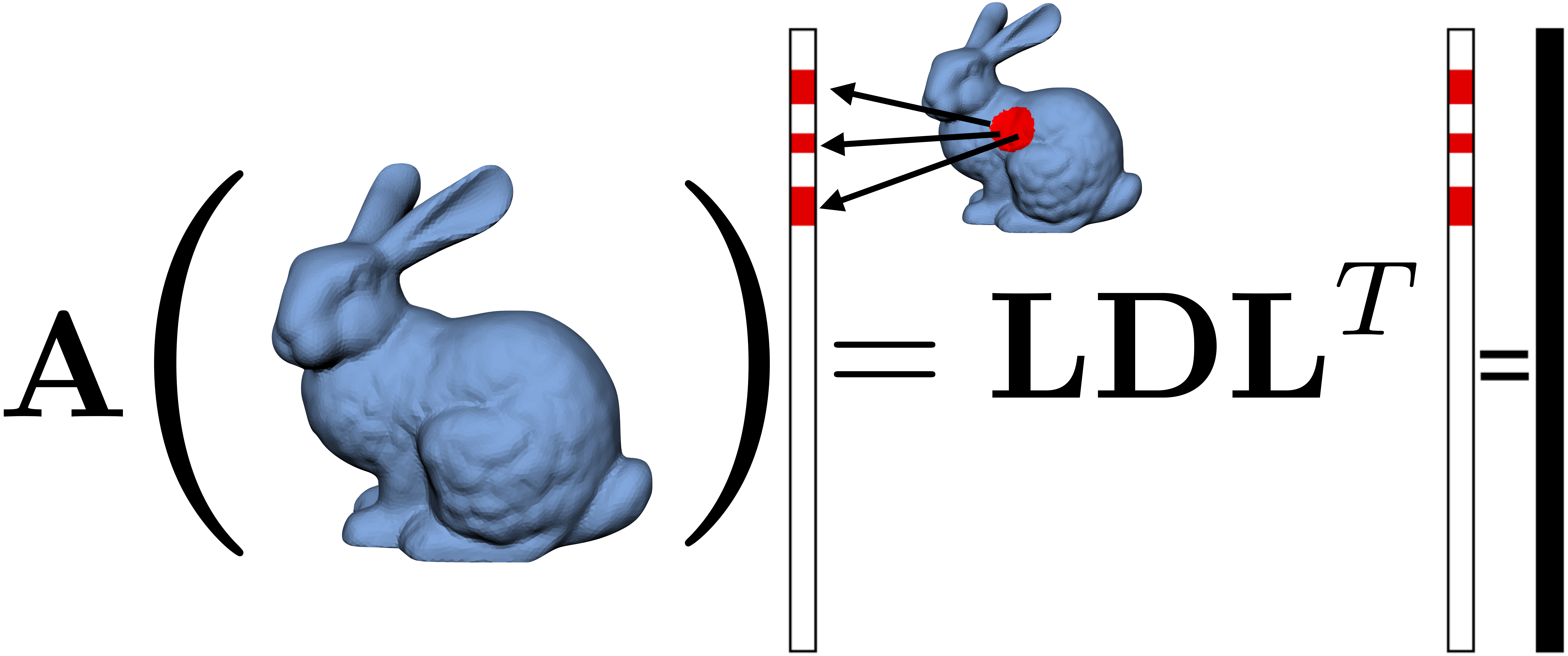
Contribution

- Idea: Reuse a global factorization: $\mathbf{A}\Bigg($  $\Bigg) = \mathbf{LDL}^T$



Contribution

- Question: Can we quickly compute subset of solution?



Sparse solution vector

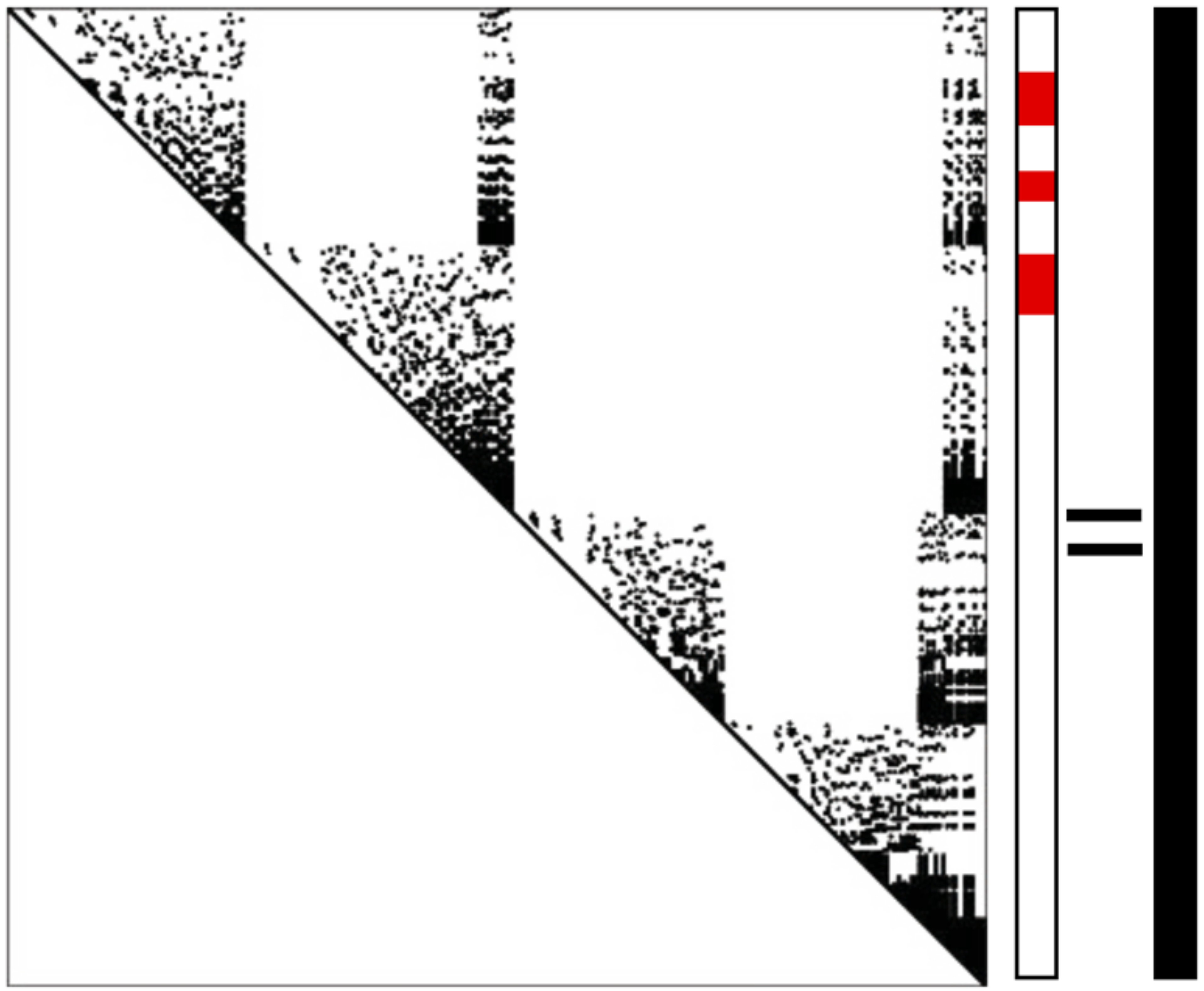
$$\mathbf{LDy} = \mathbf{b} \ (1) \text{Forward solve}$$

$$\mathbf{L}^T\mathbf{x} = \mathbf{y} \ (2) \text{Back solve}$$

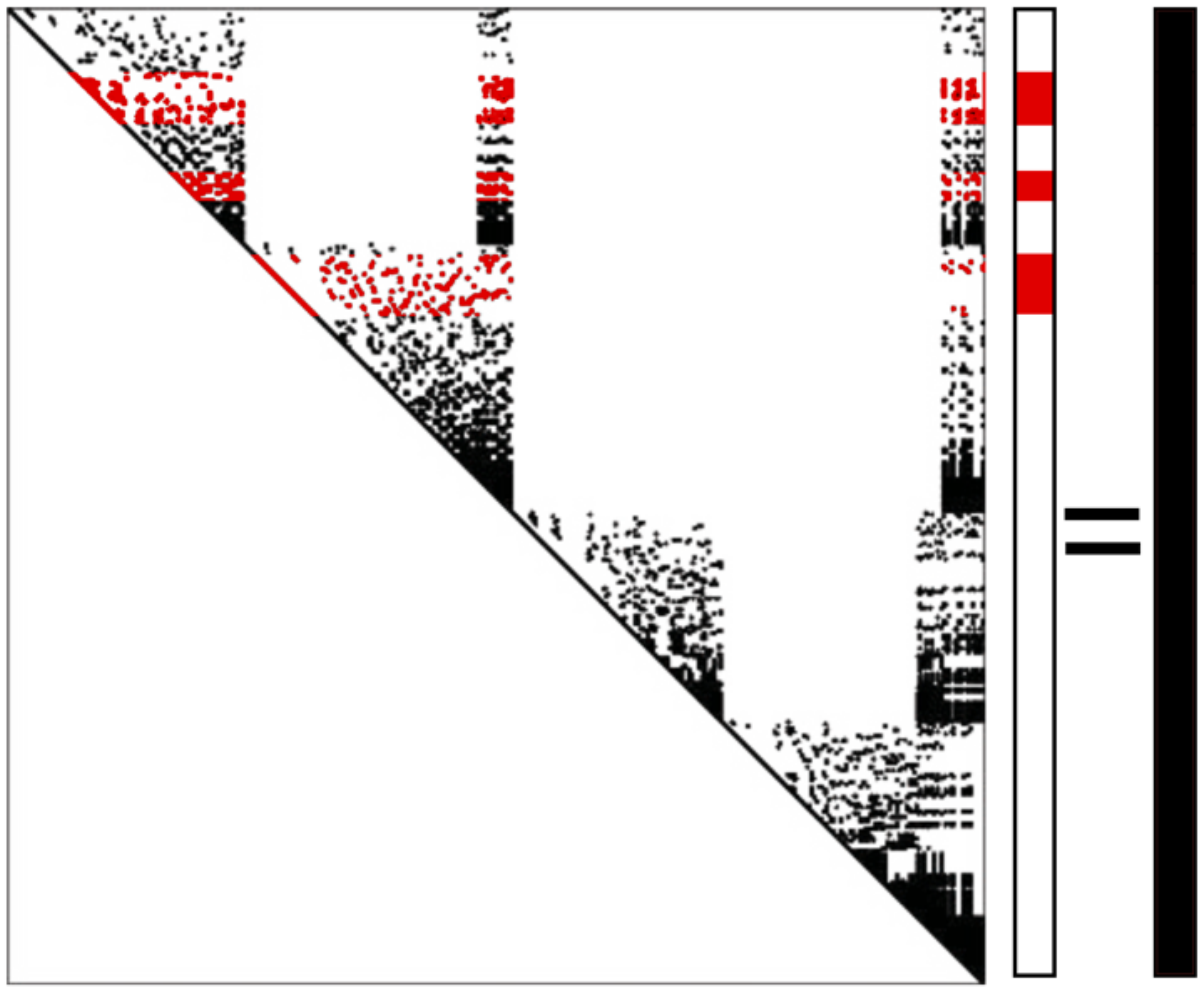
- $\mathbf{x}$ will generally be dense.

- **Central insight:** If we are interested in only a subset of values in $\mathbf{x}$ we do not have to compute all values during the back solve!

Sparse solution vector

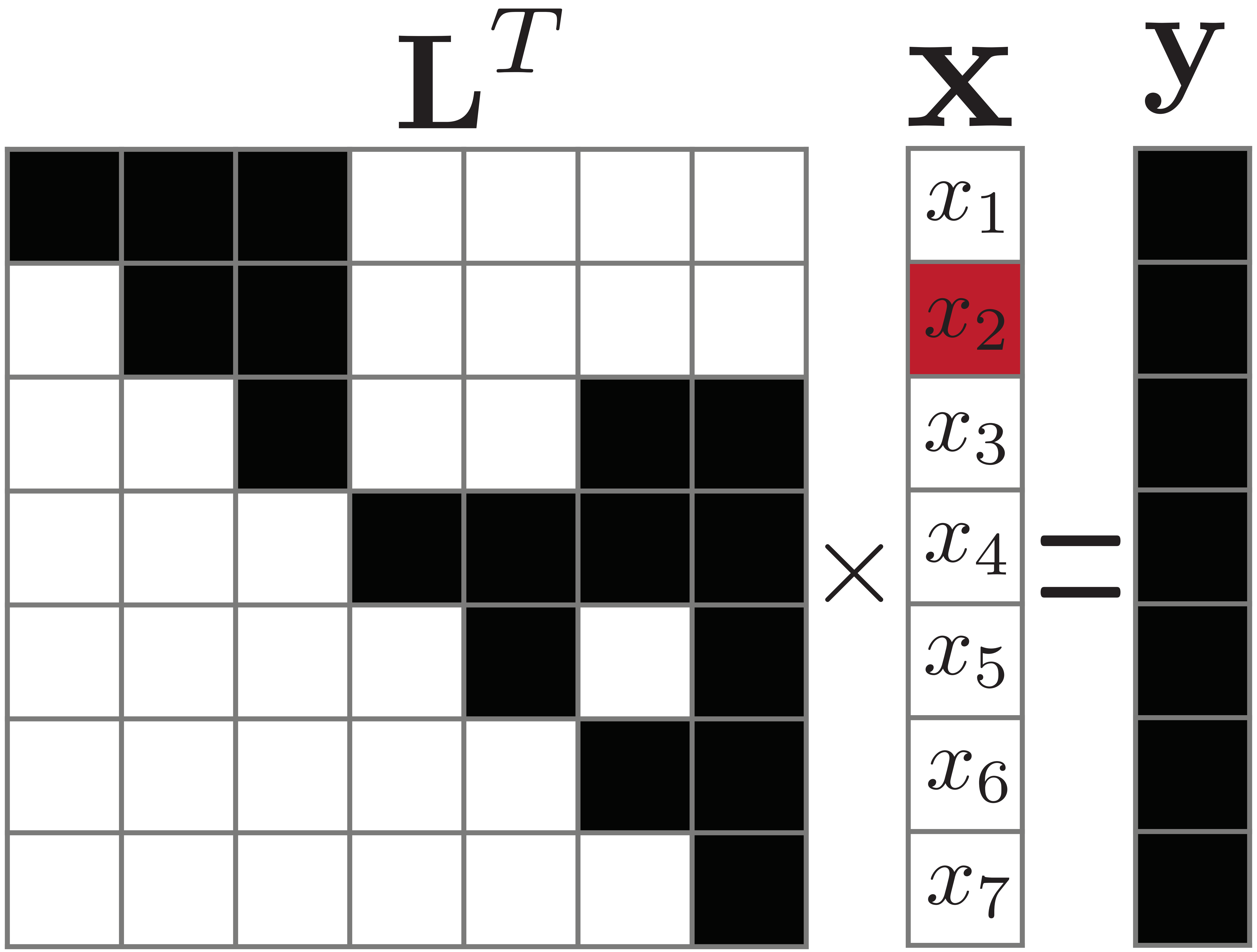


Sparse solution vector

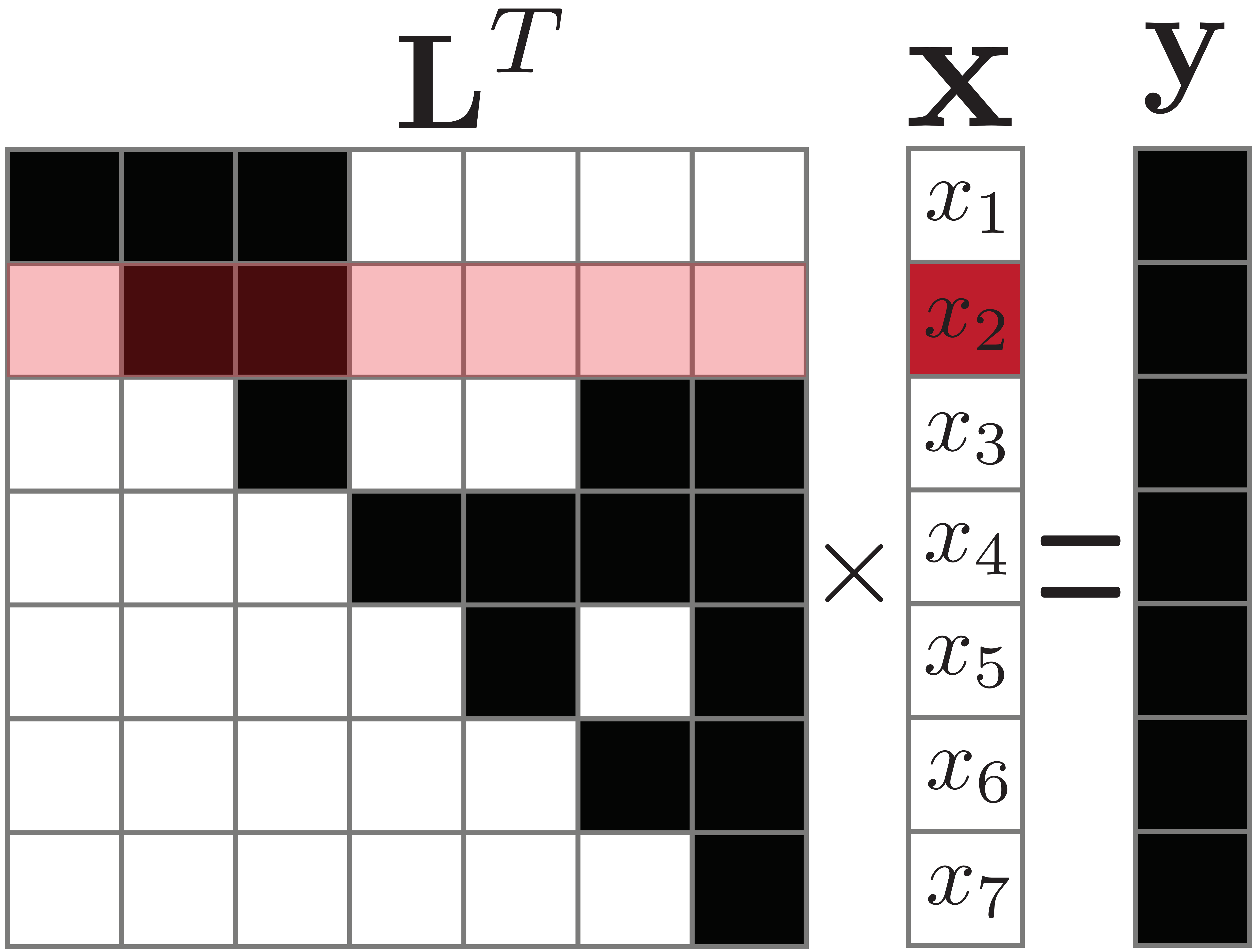


- How many values do we have to compute additionally?
- Can we identify them efficiently?

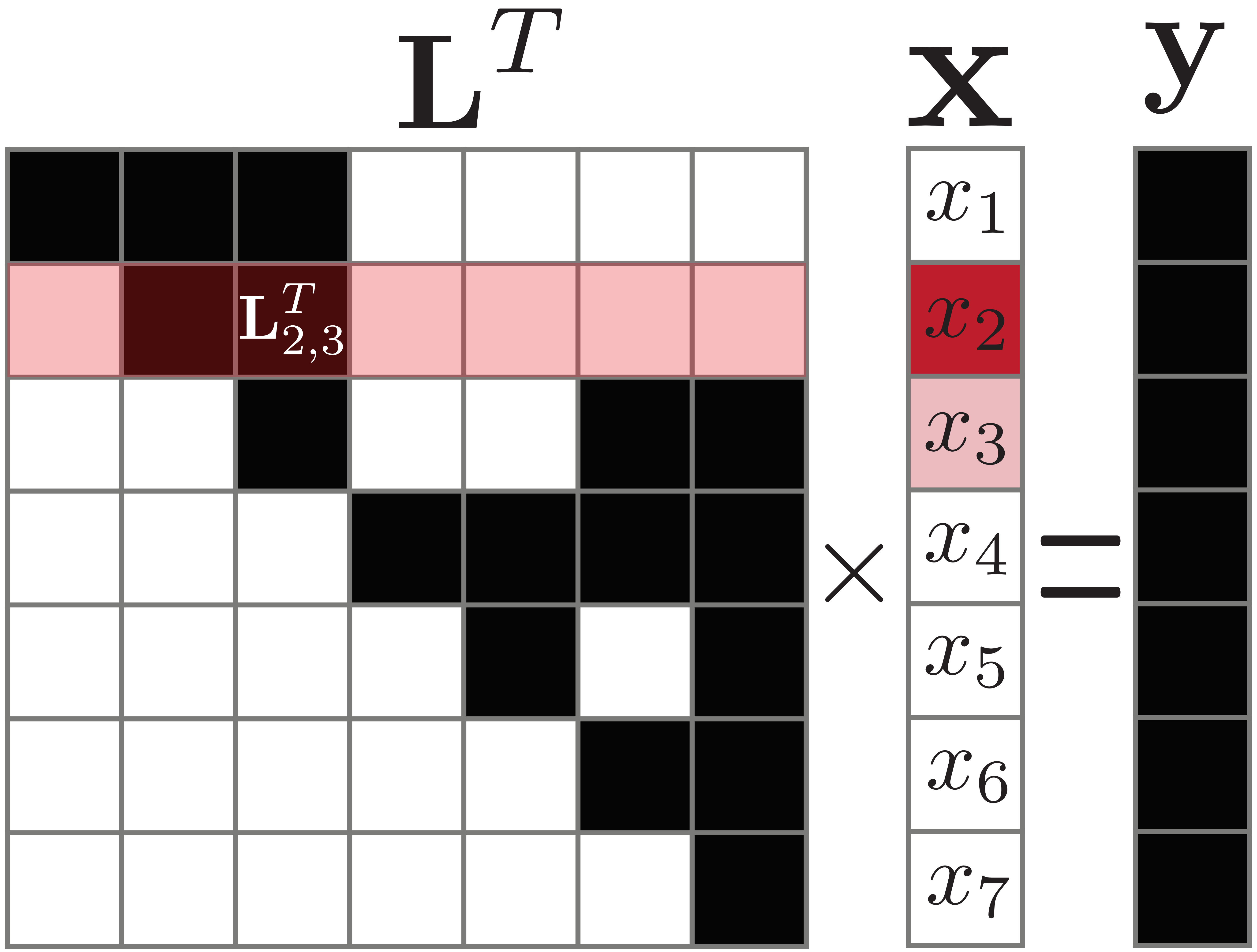
Sparse back-solve



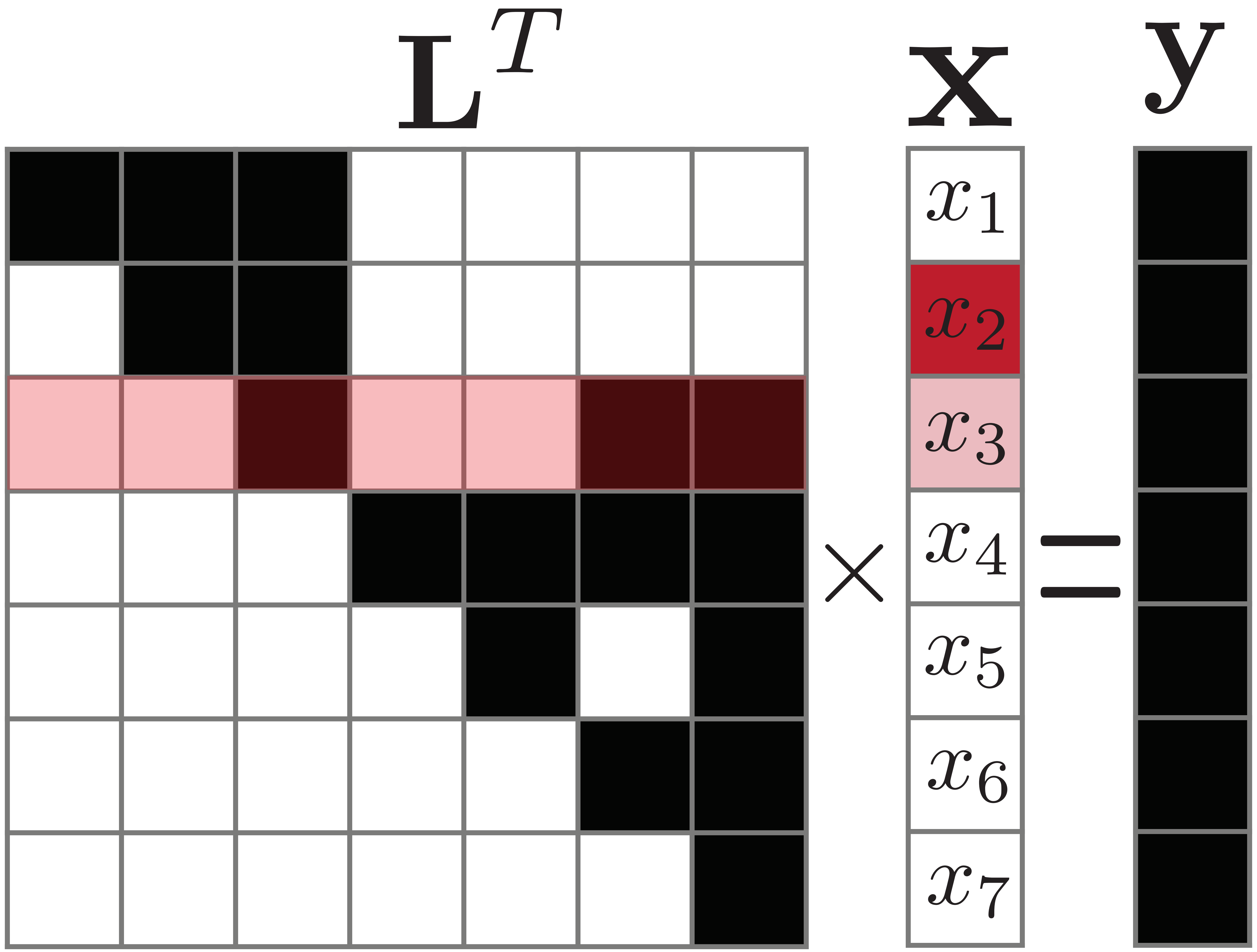
Sparse back-solve



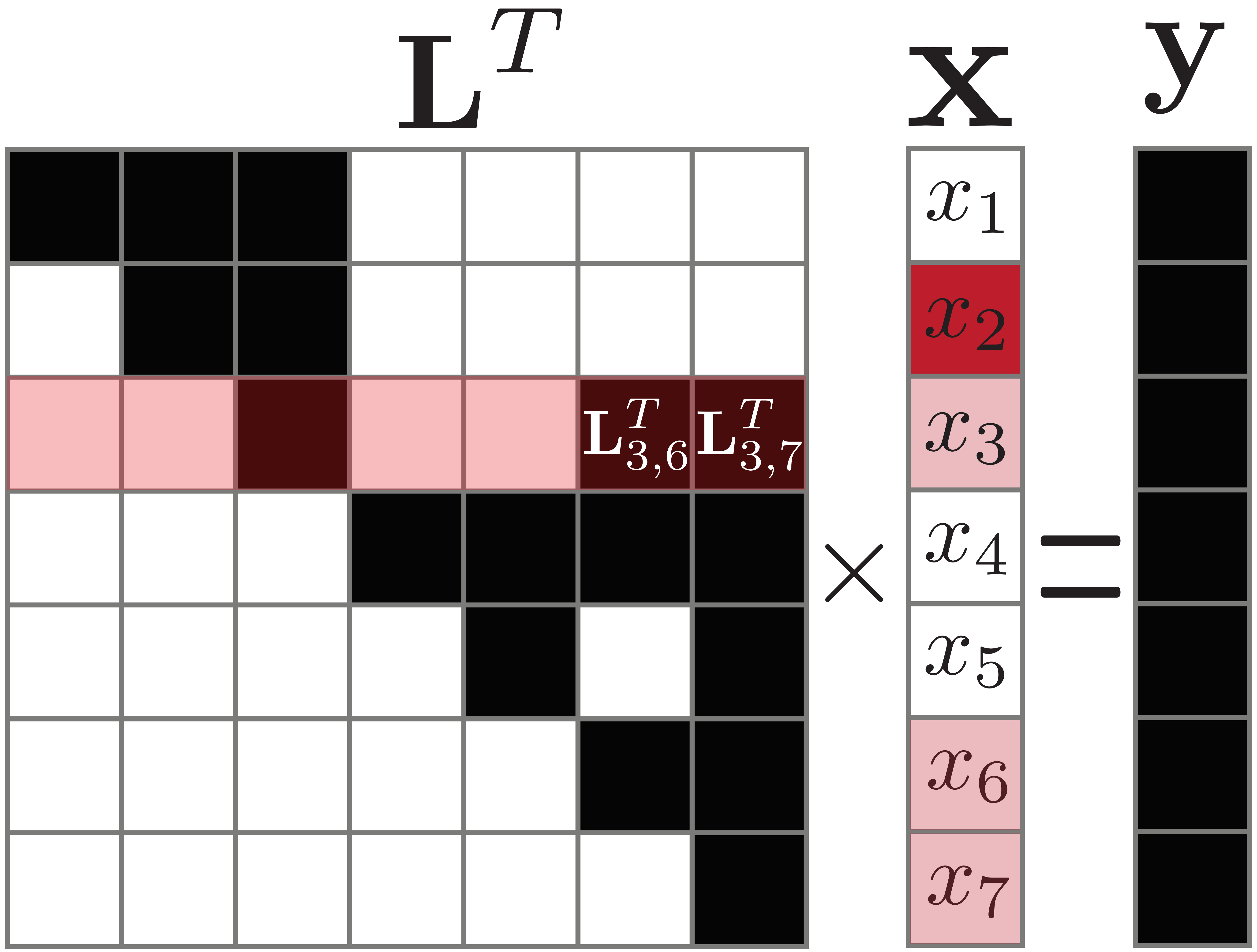
Sparse back-solve



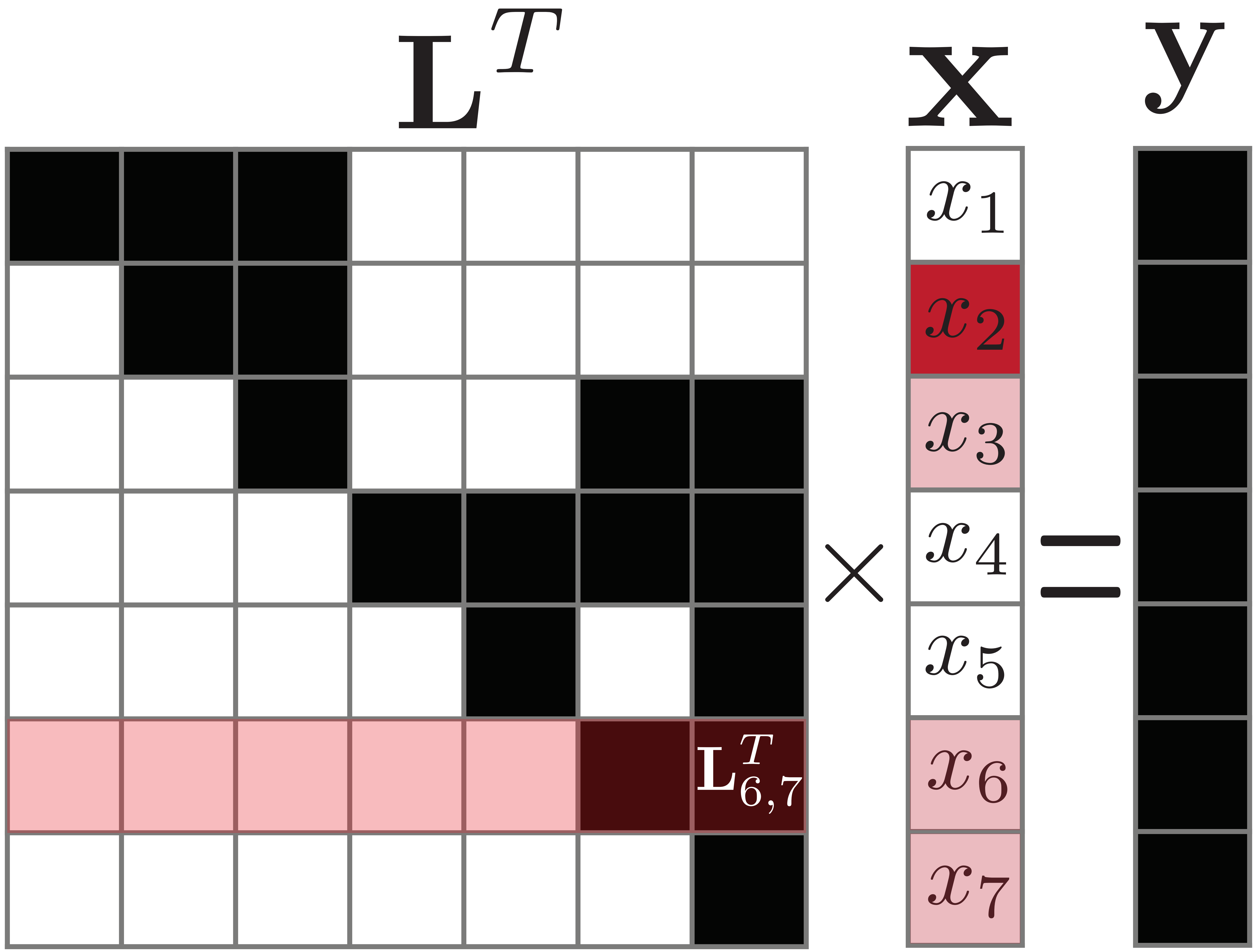
Sparse back-solve



Sparse back-solve



Sparse back-solve



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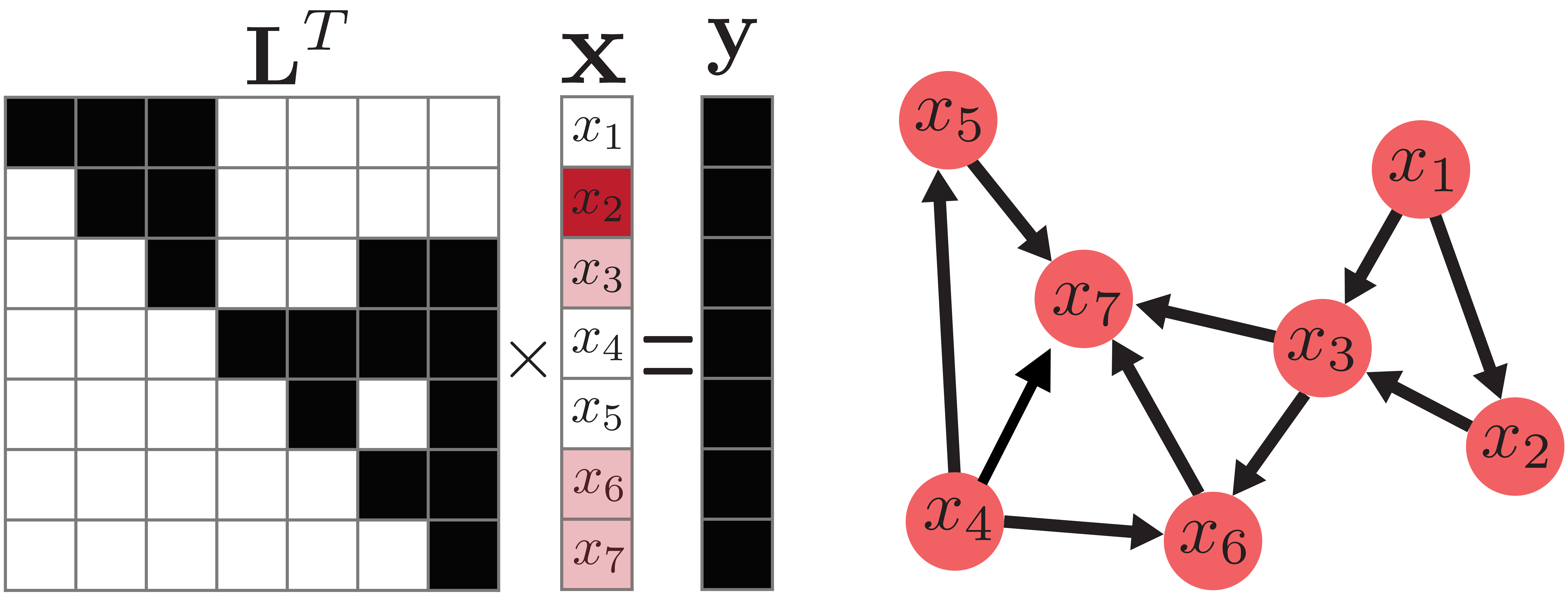

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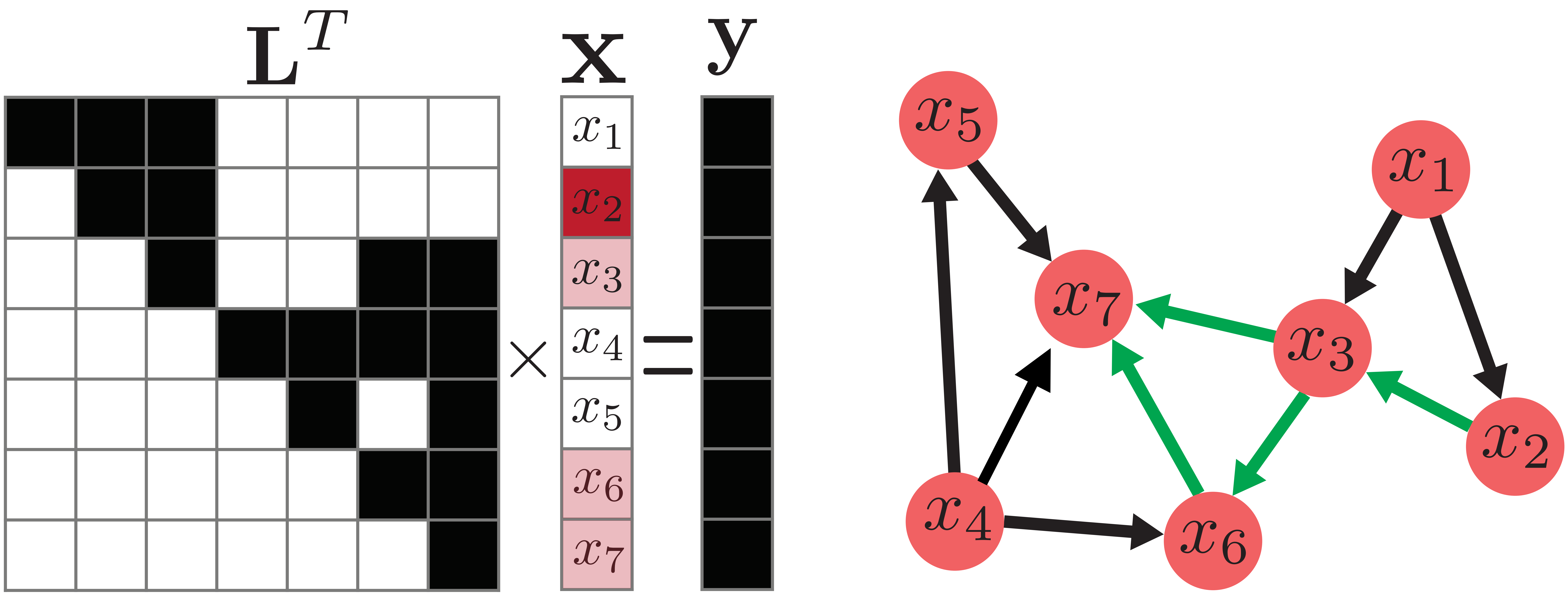
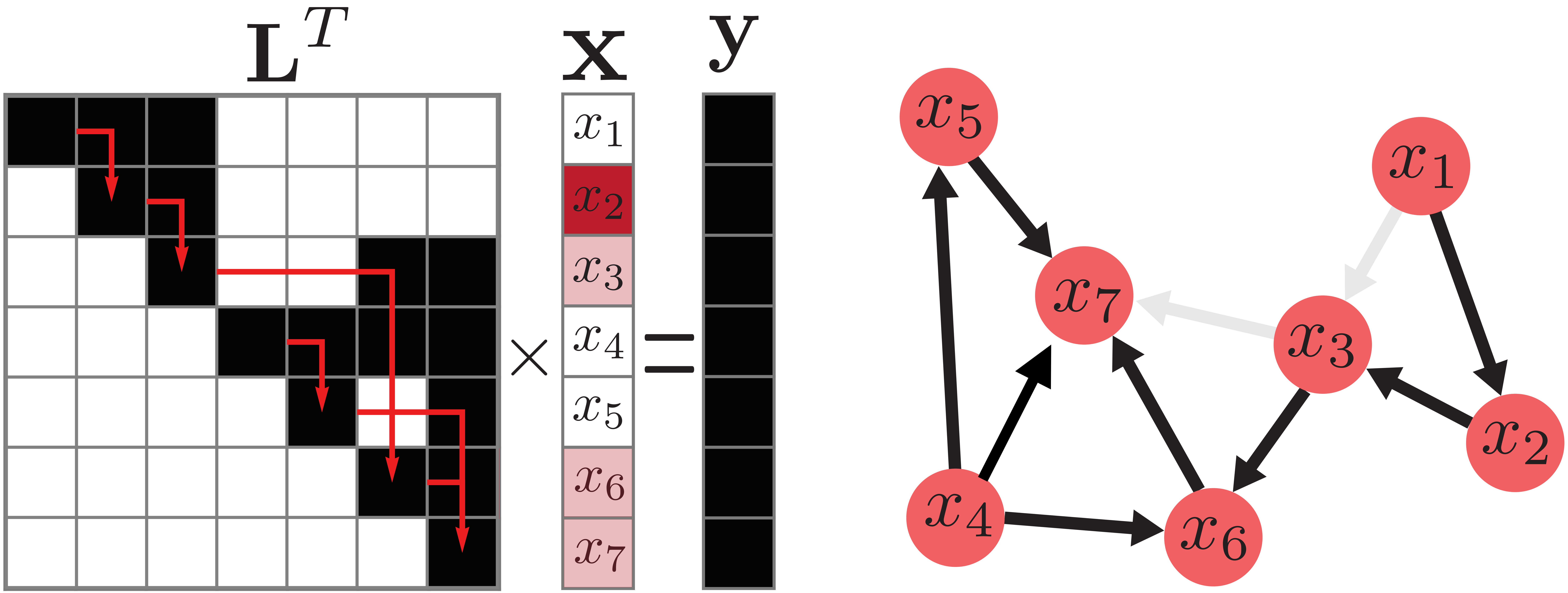


Sparse back-solve



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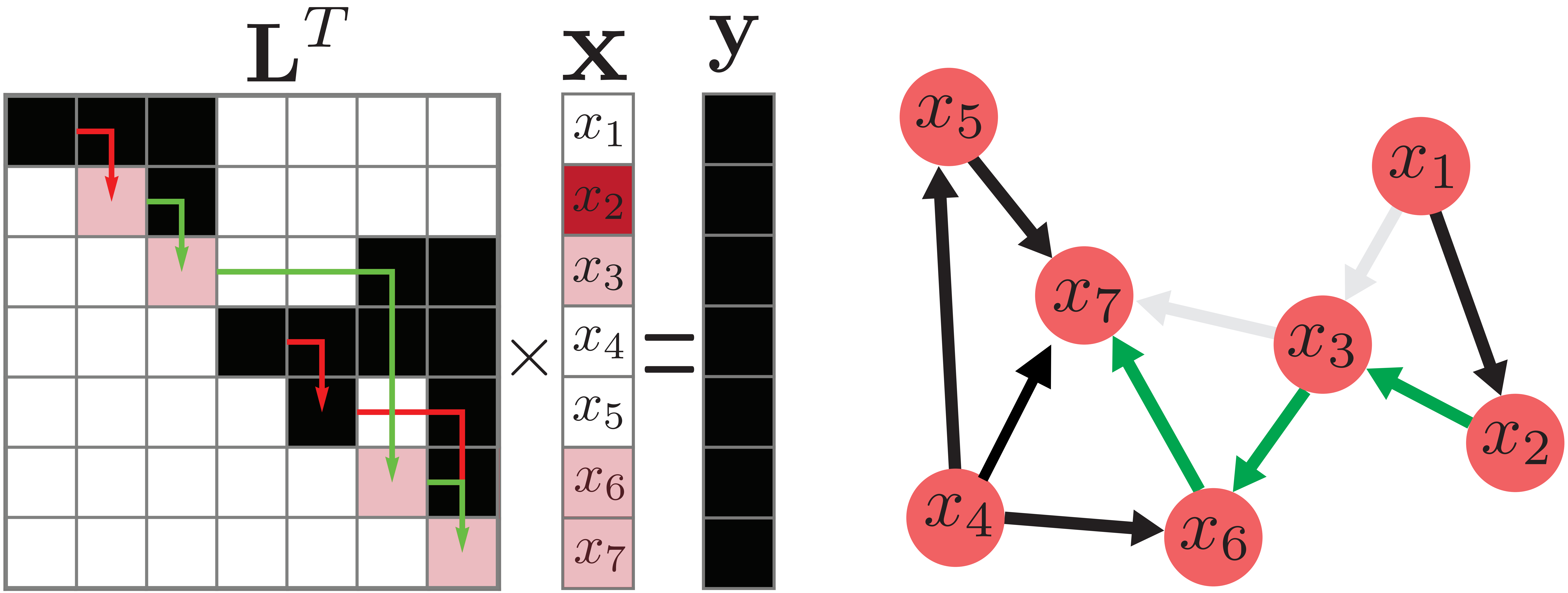
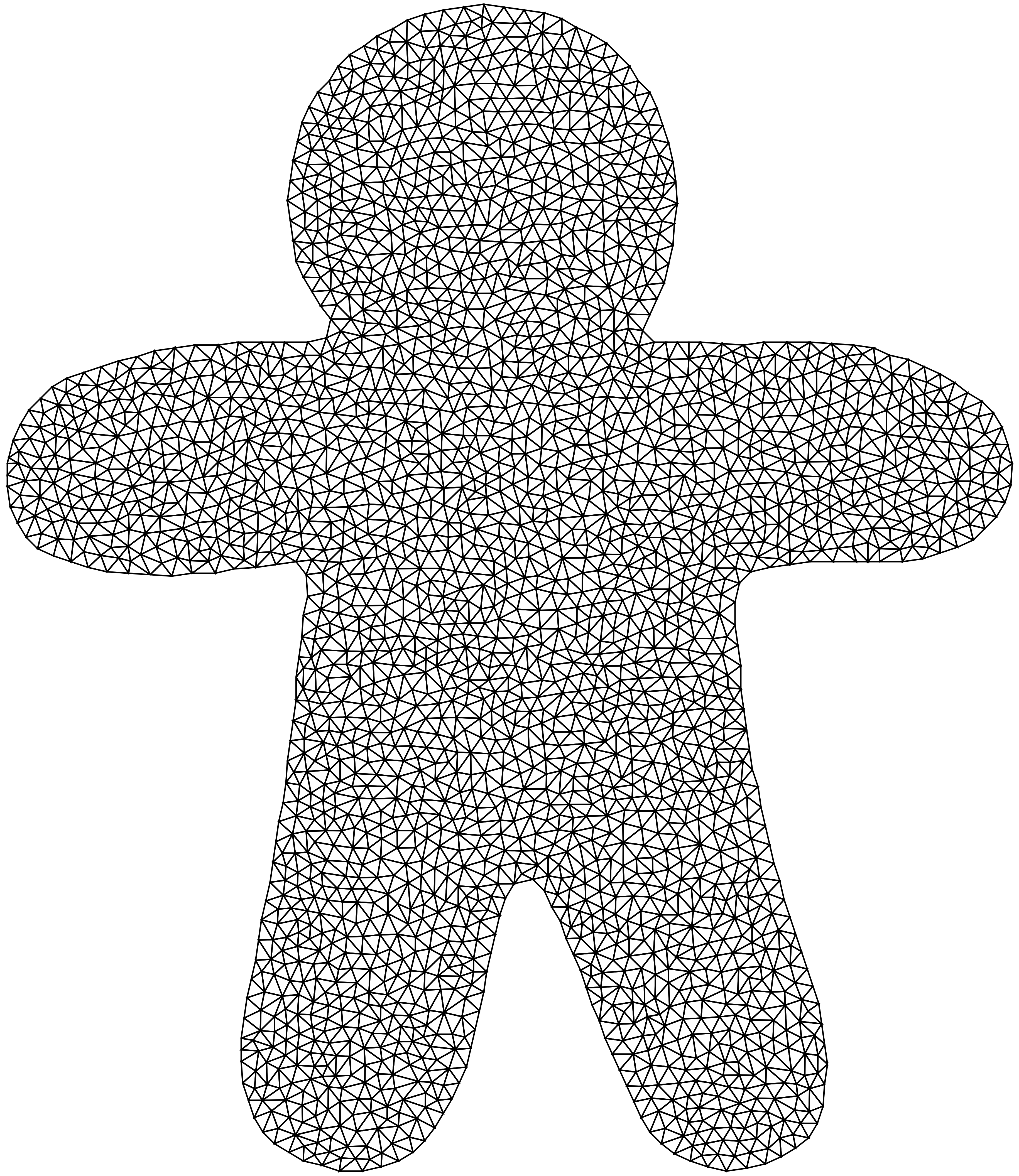


Nested dissection reordering

Input mesh



Ordered Laplacian

Nested dissection reordering

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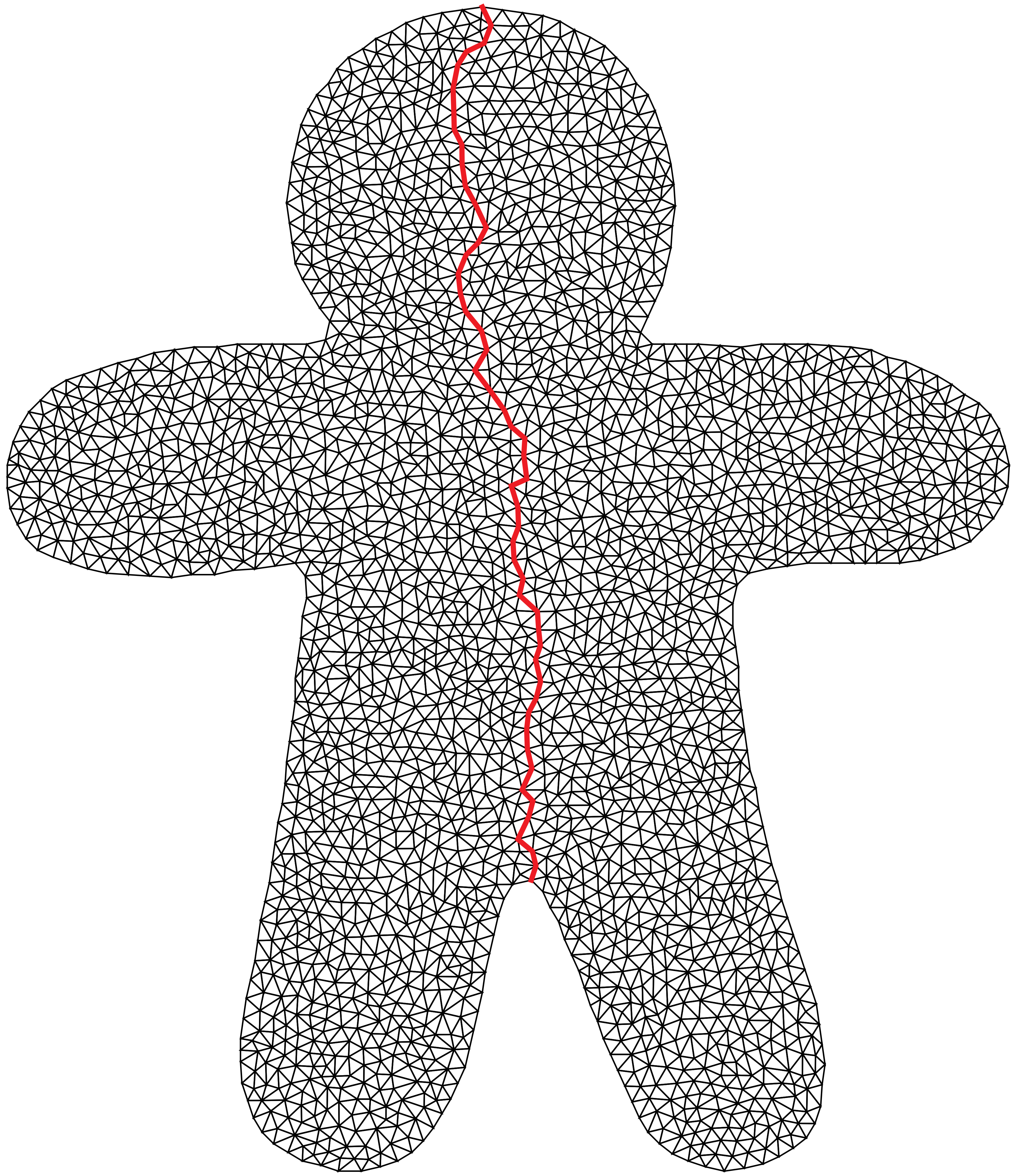

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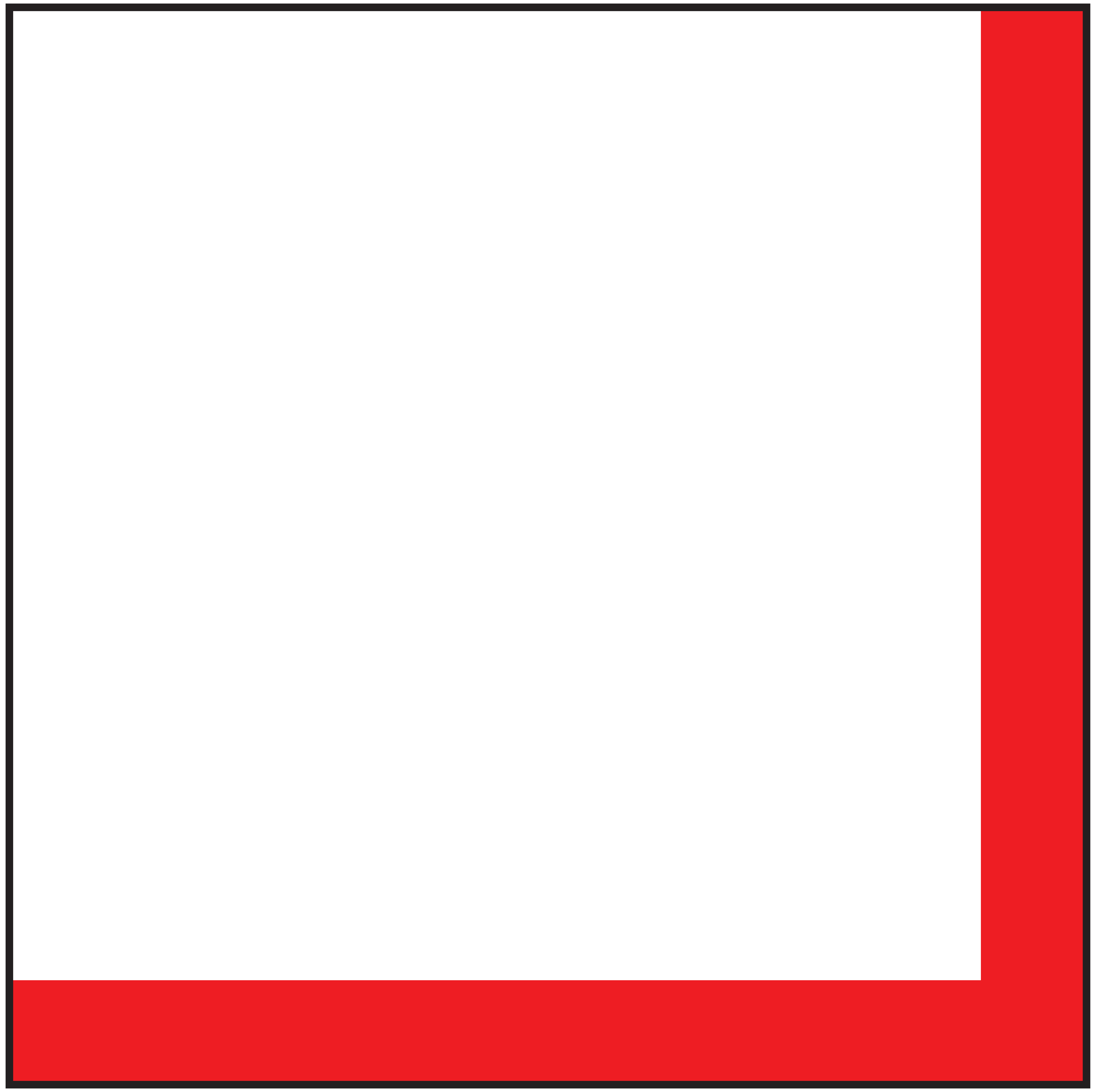

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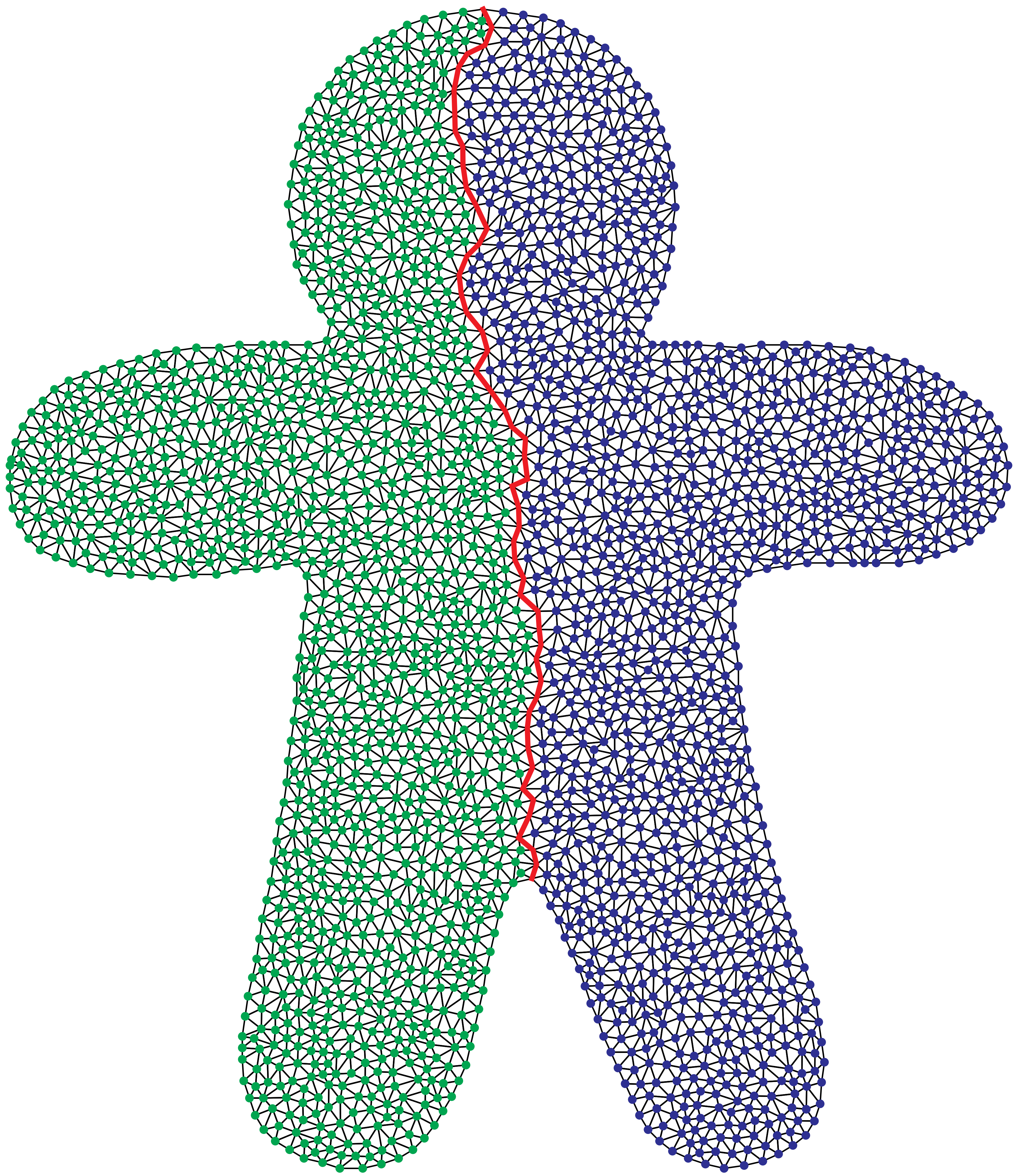

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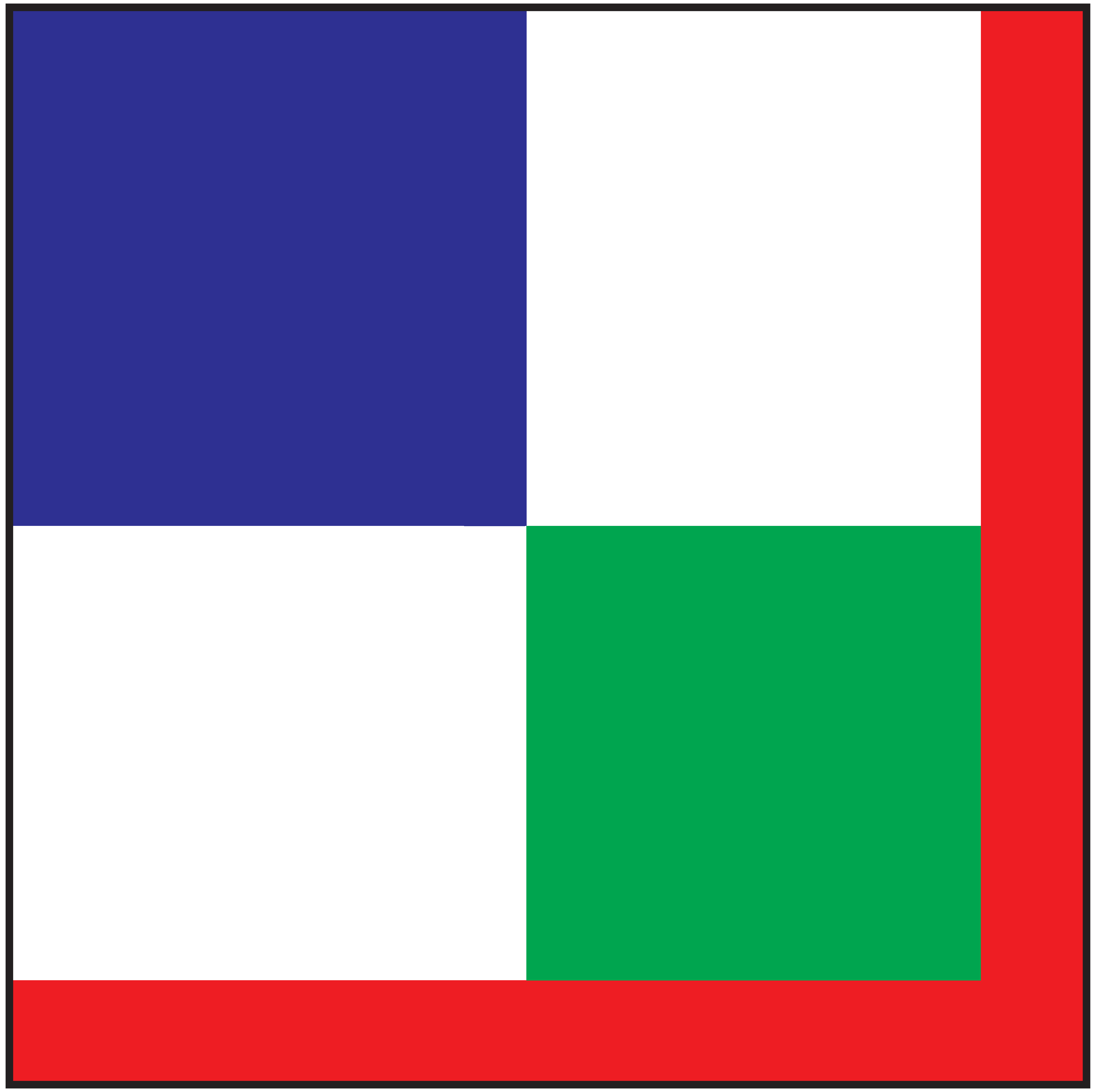

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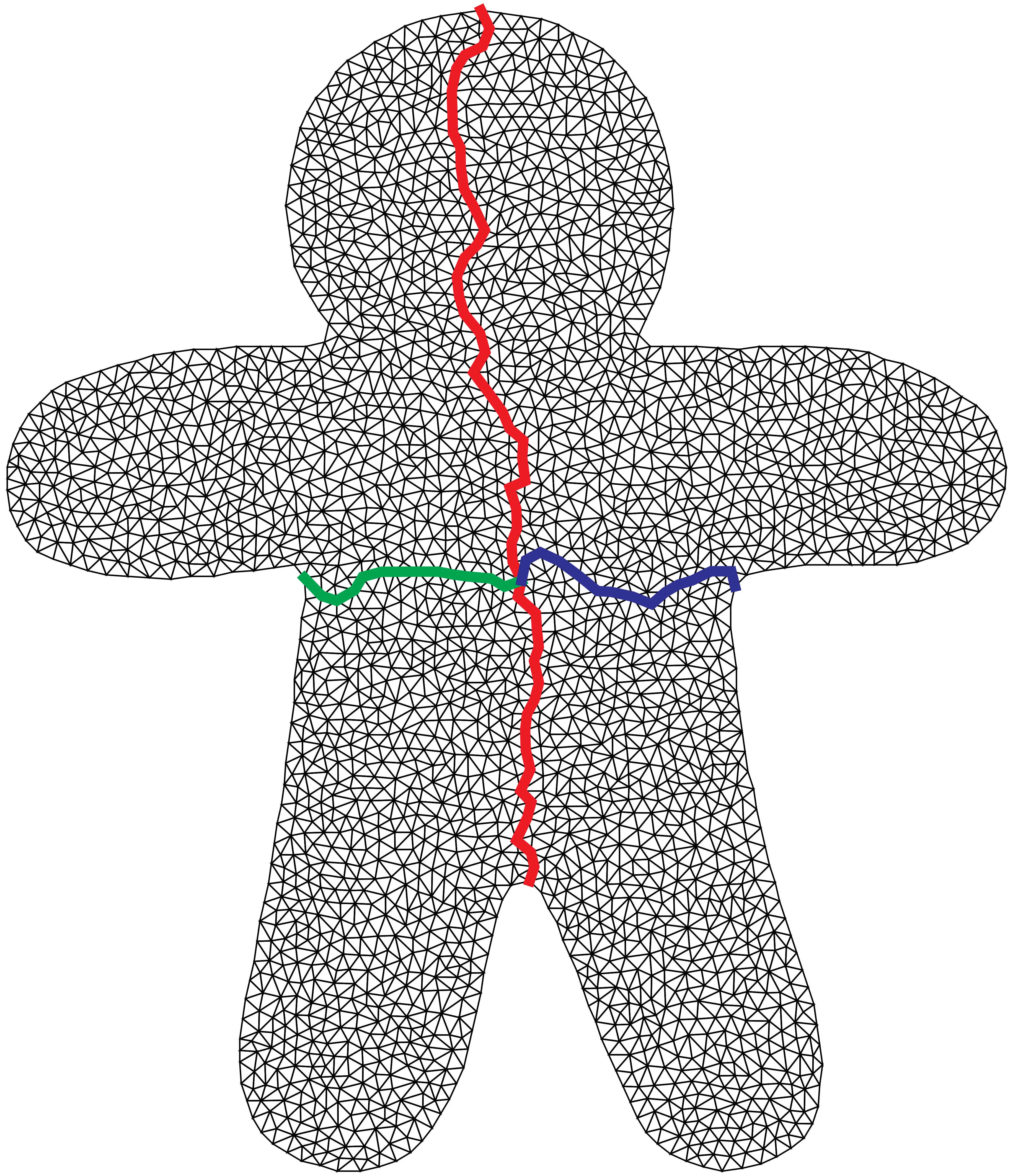

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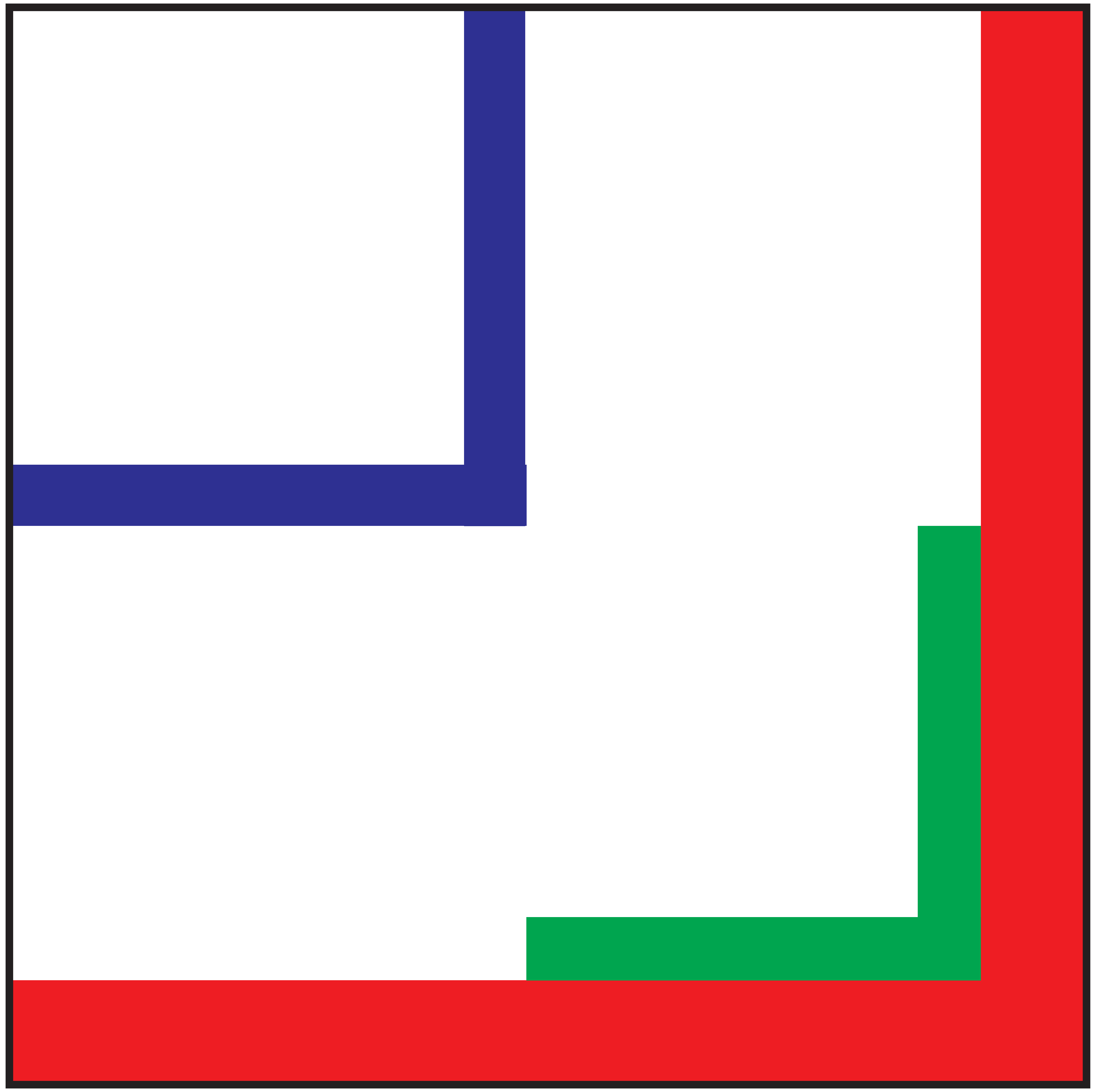

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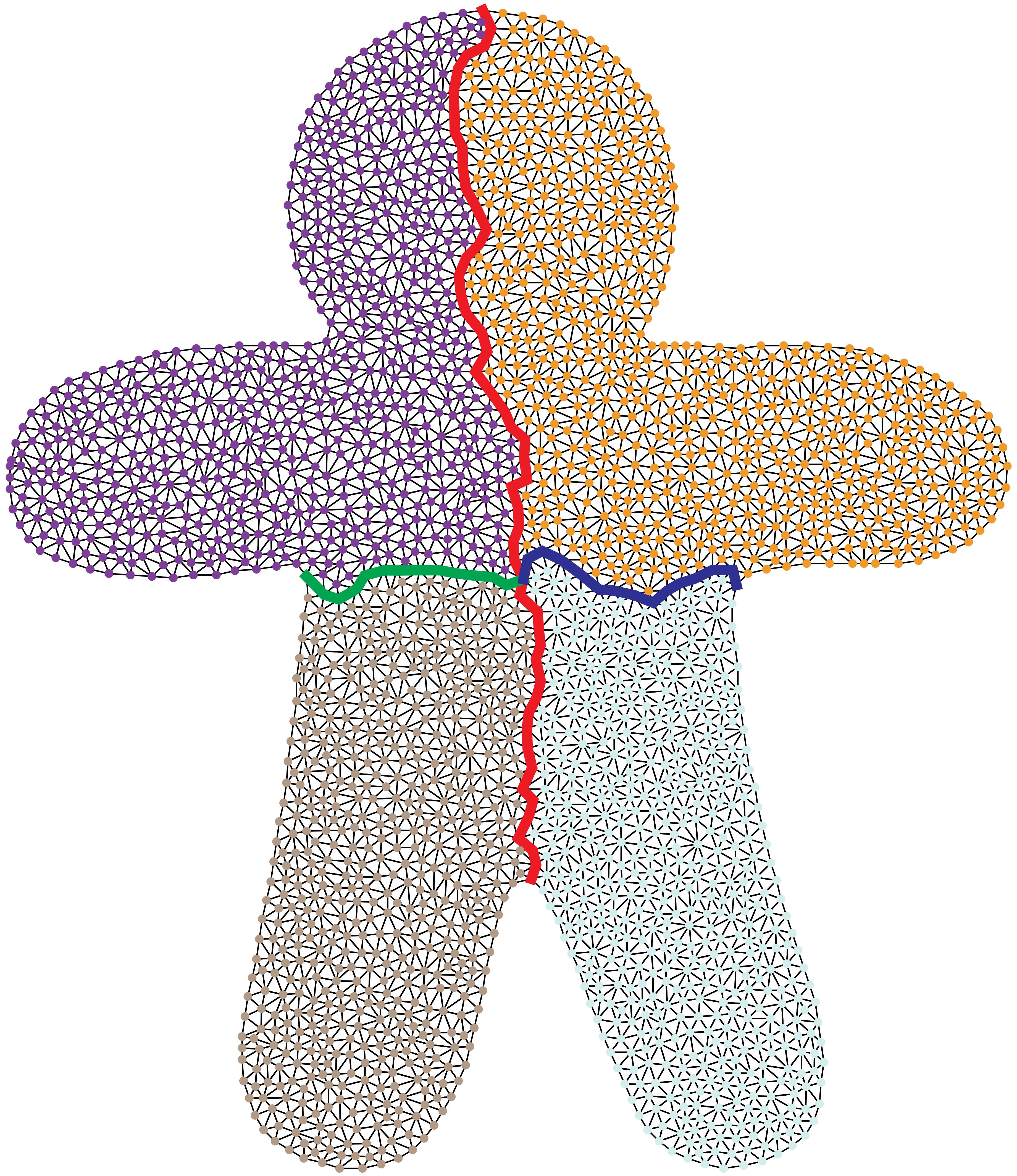

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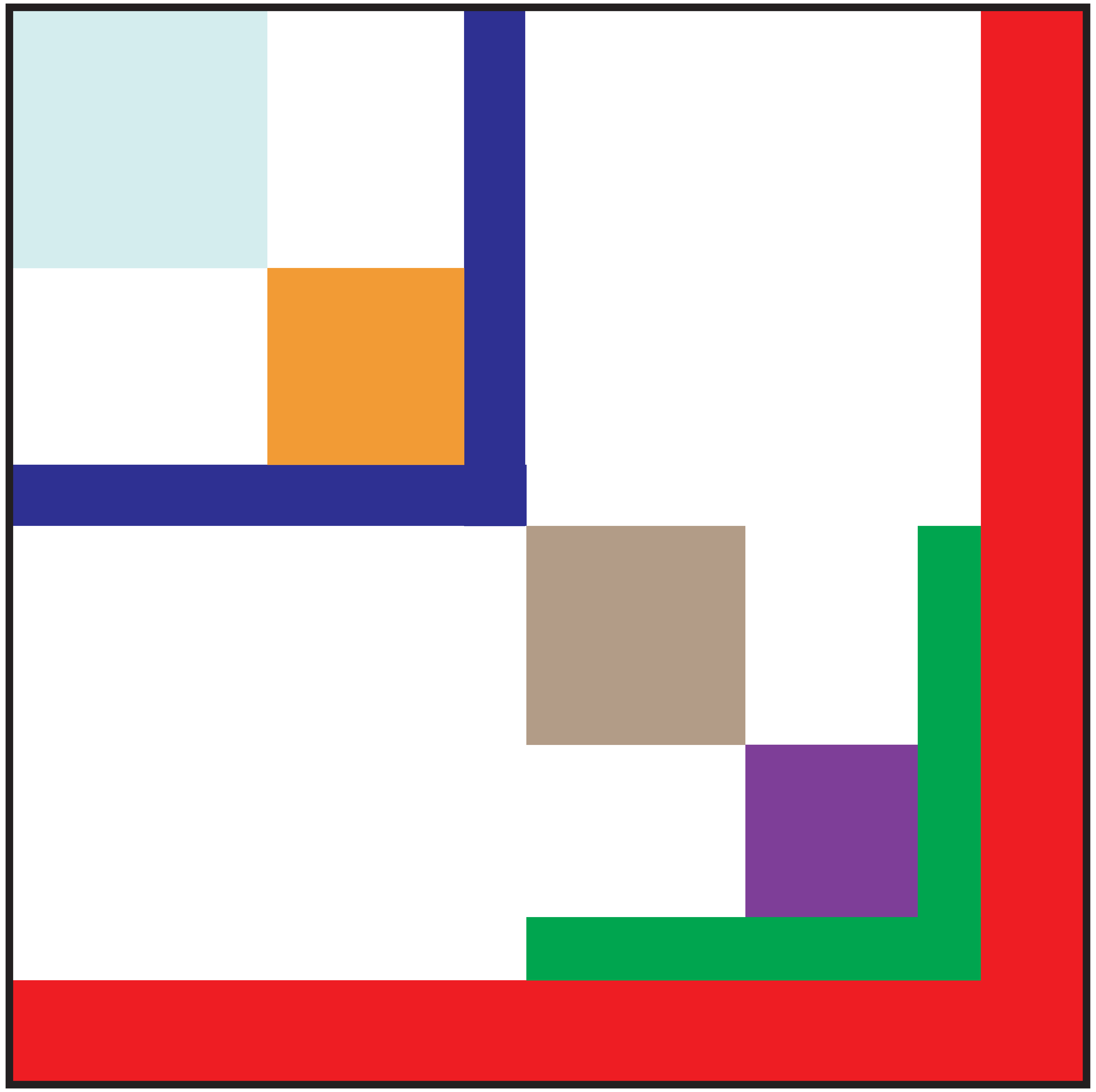
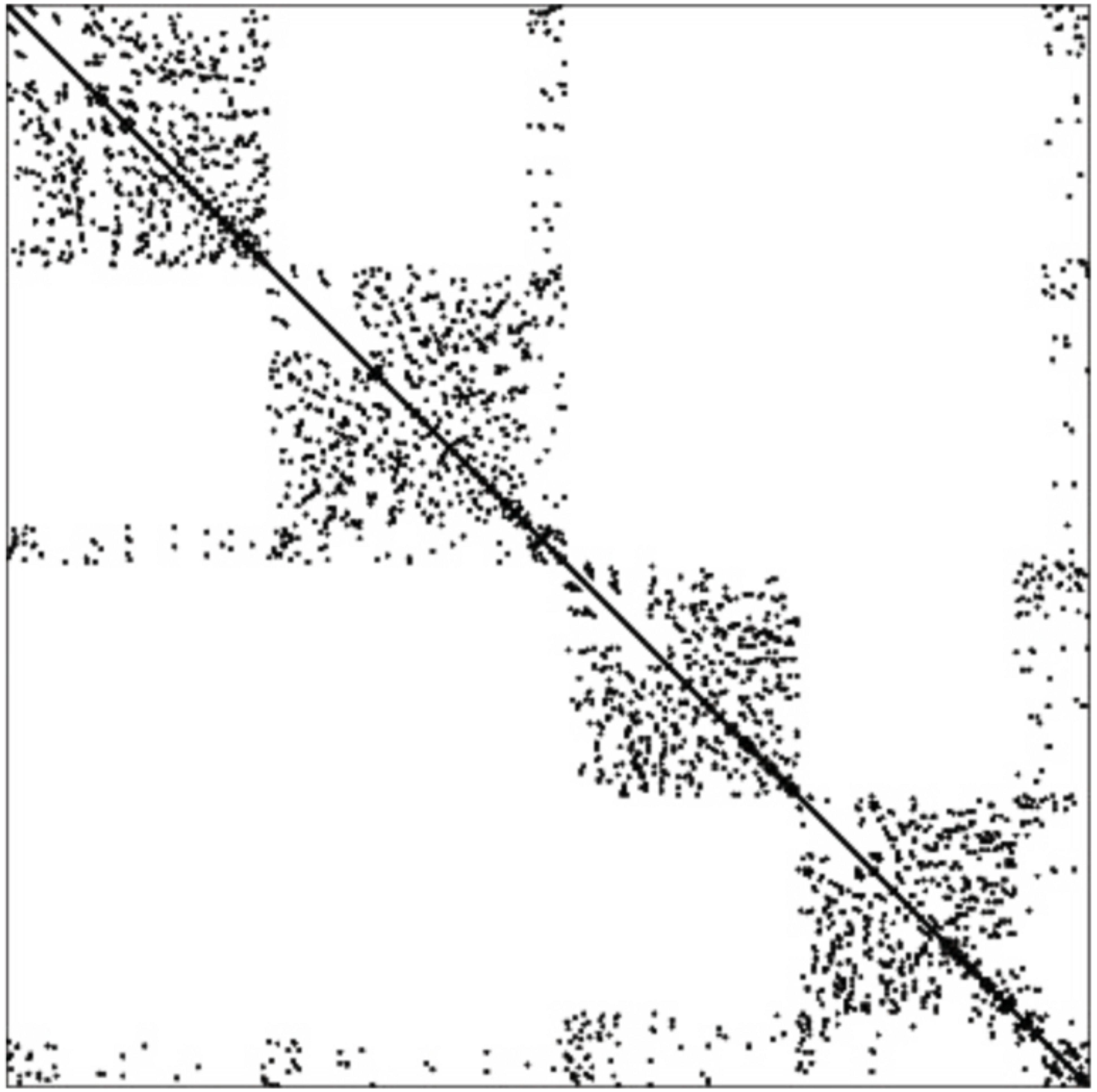
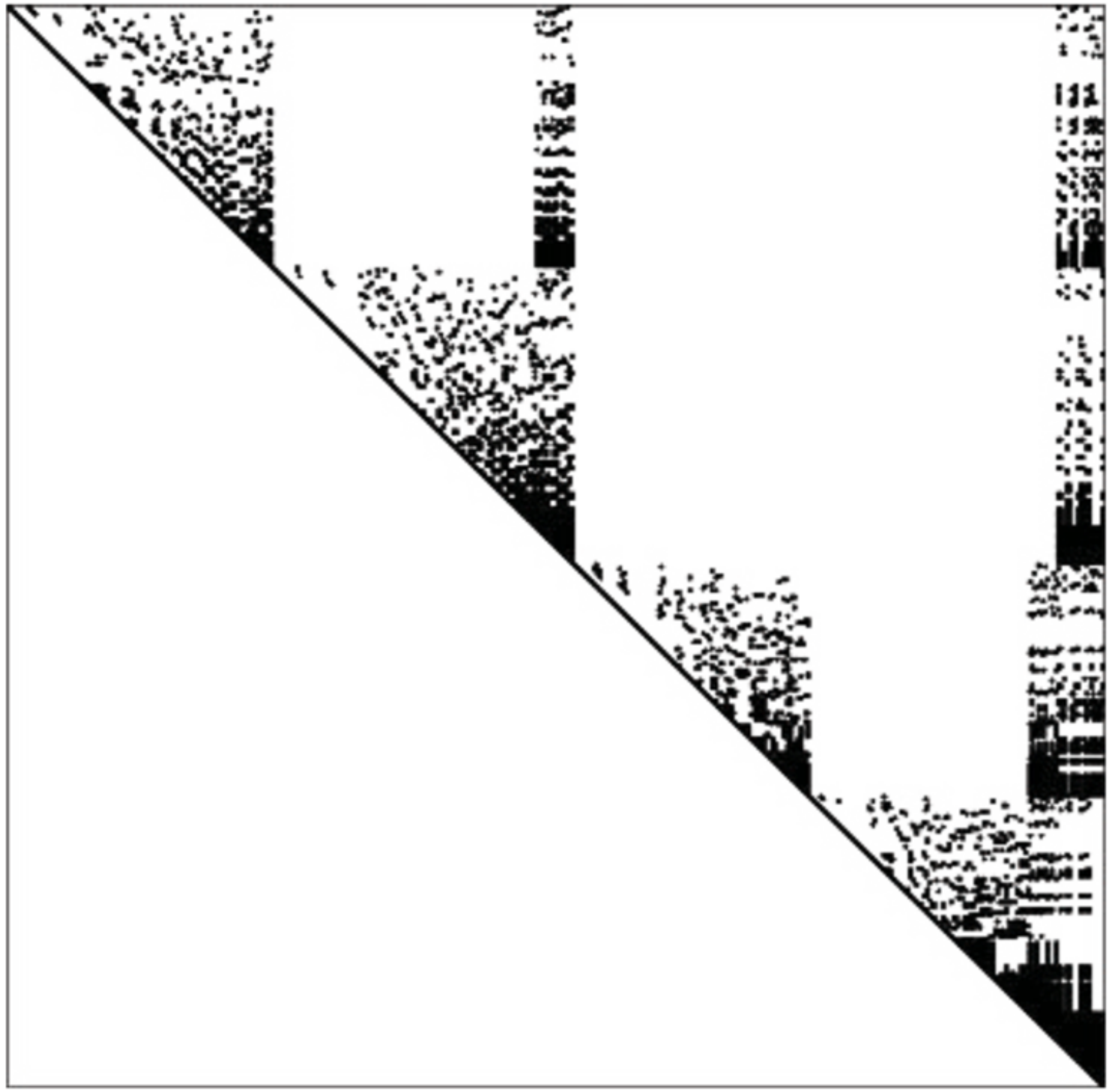


Nested dissection reordering

Input matrix

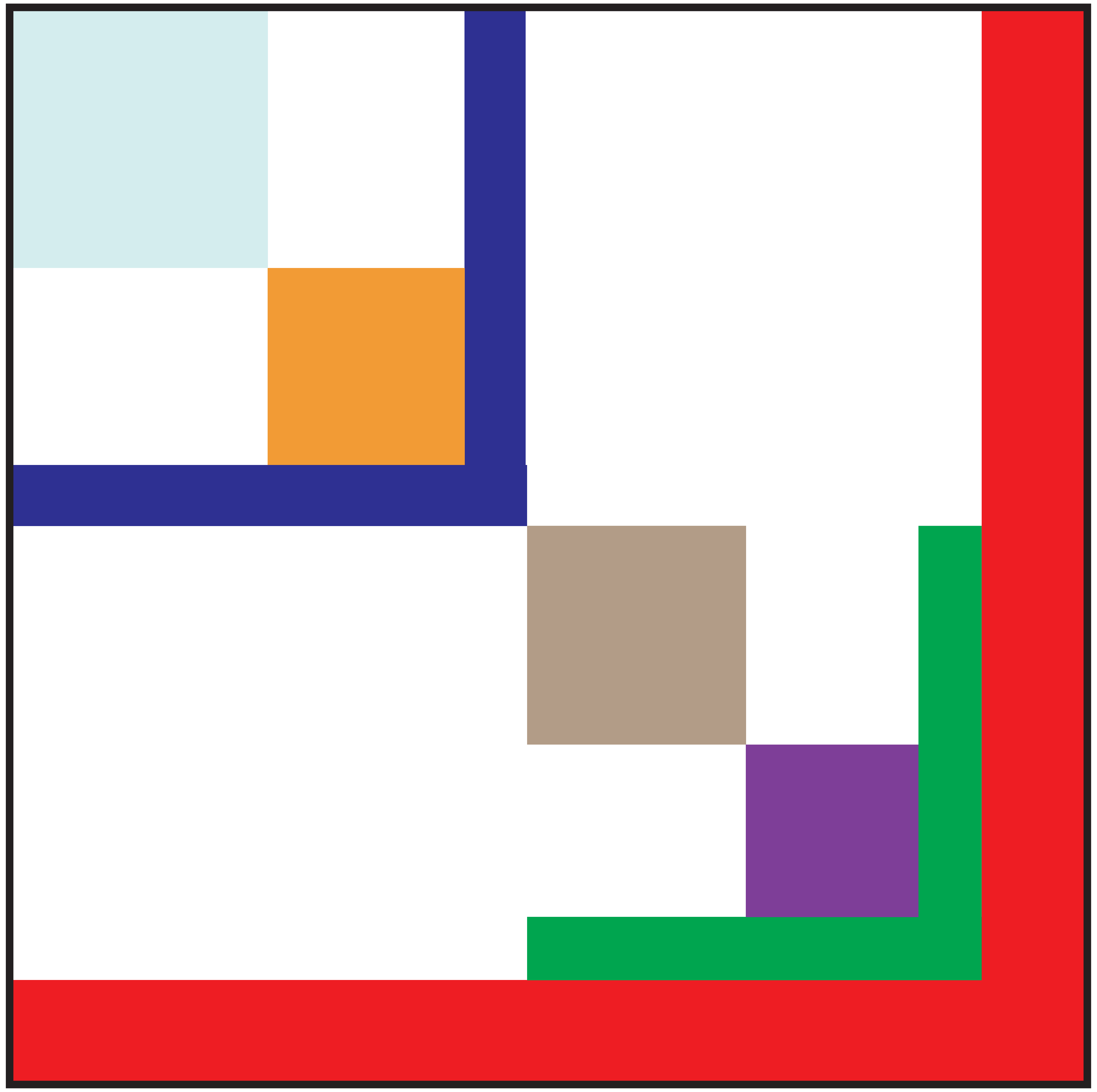


Cholesky factor $\mathbf{L}^T$

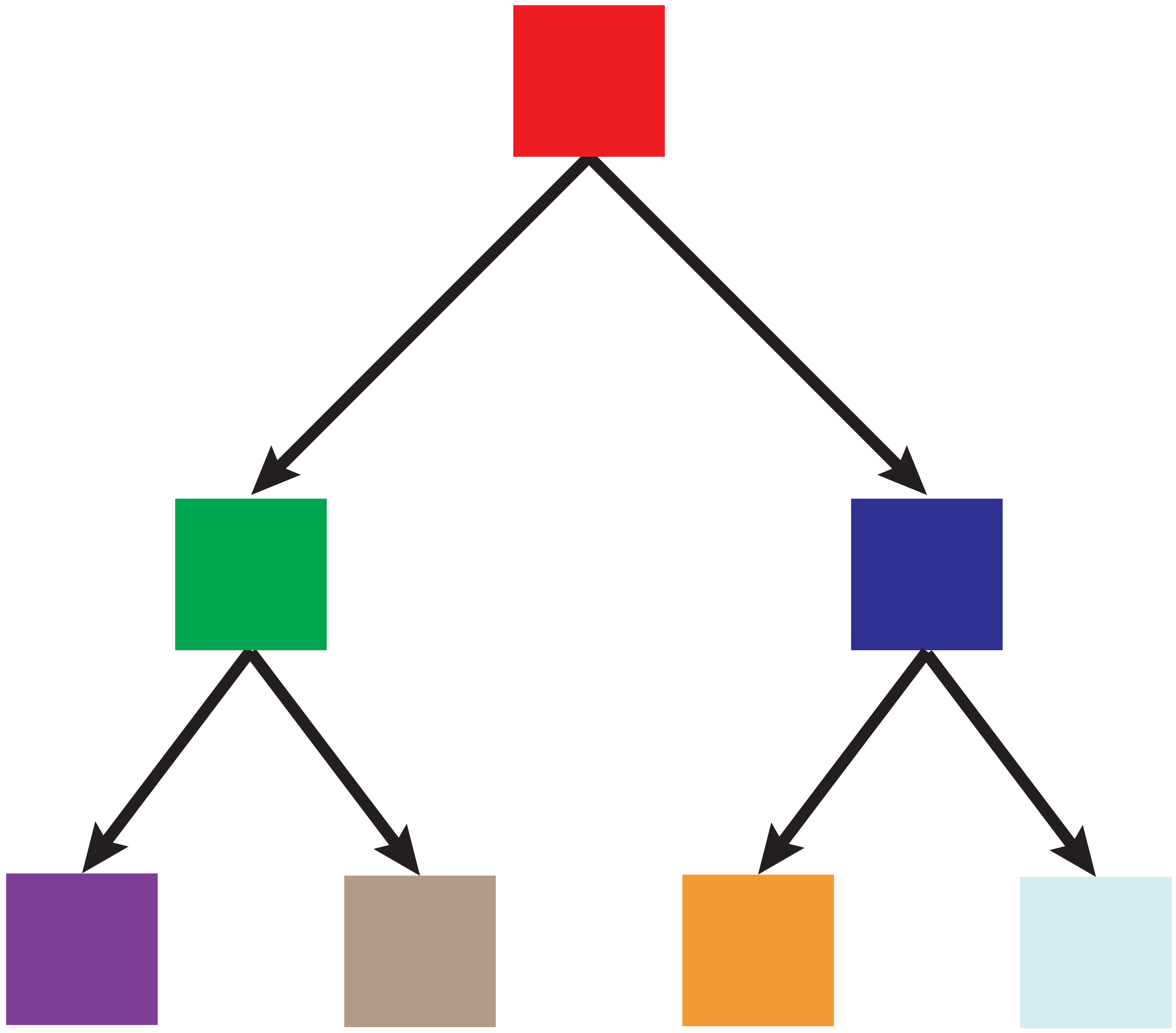


Nested dissection reordering

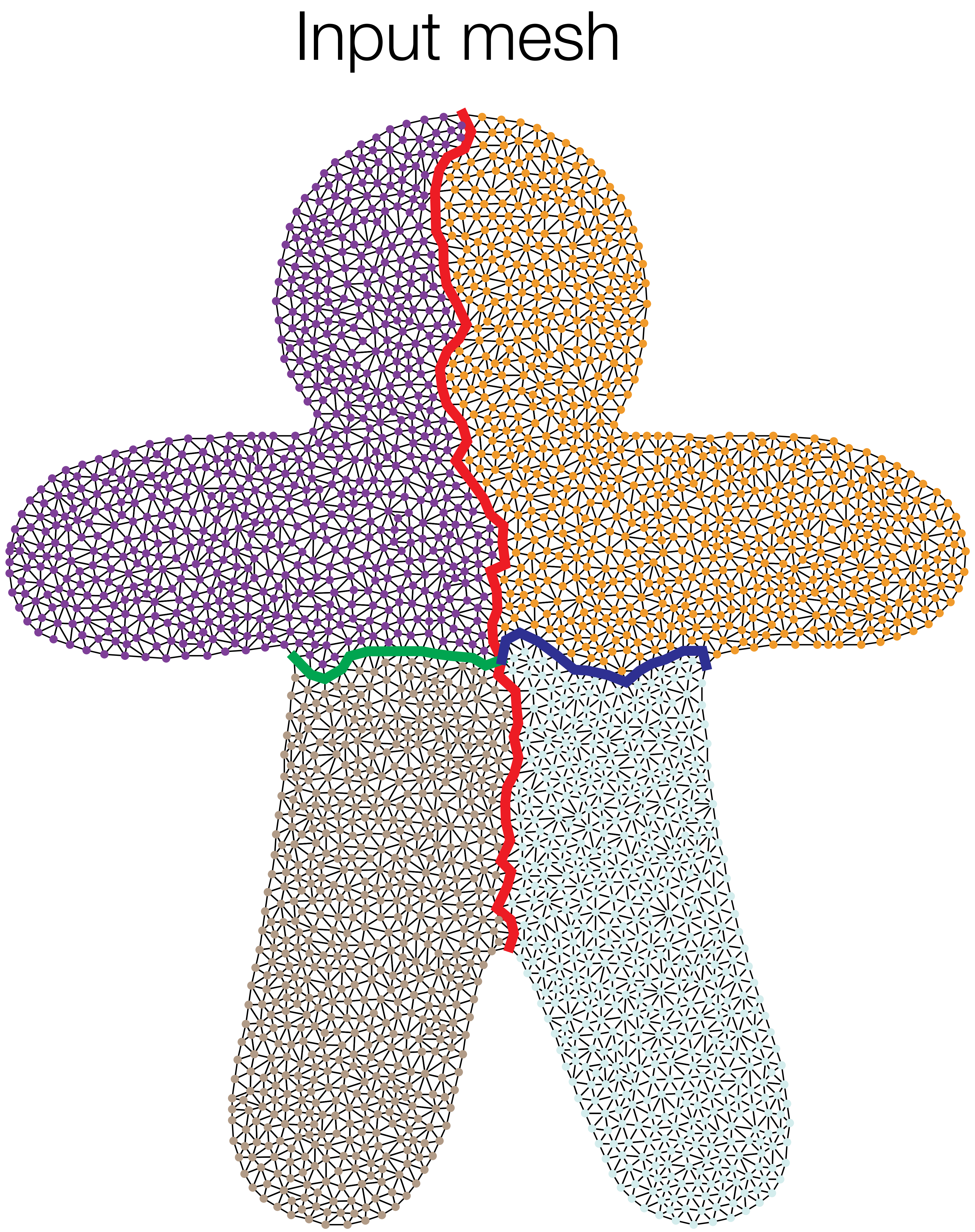
Cholesky factor

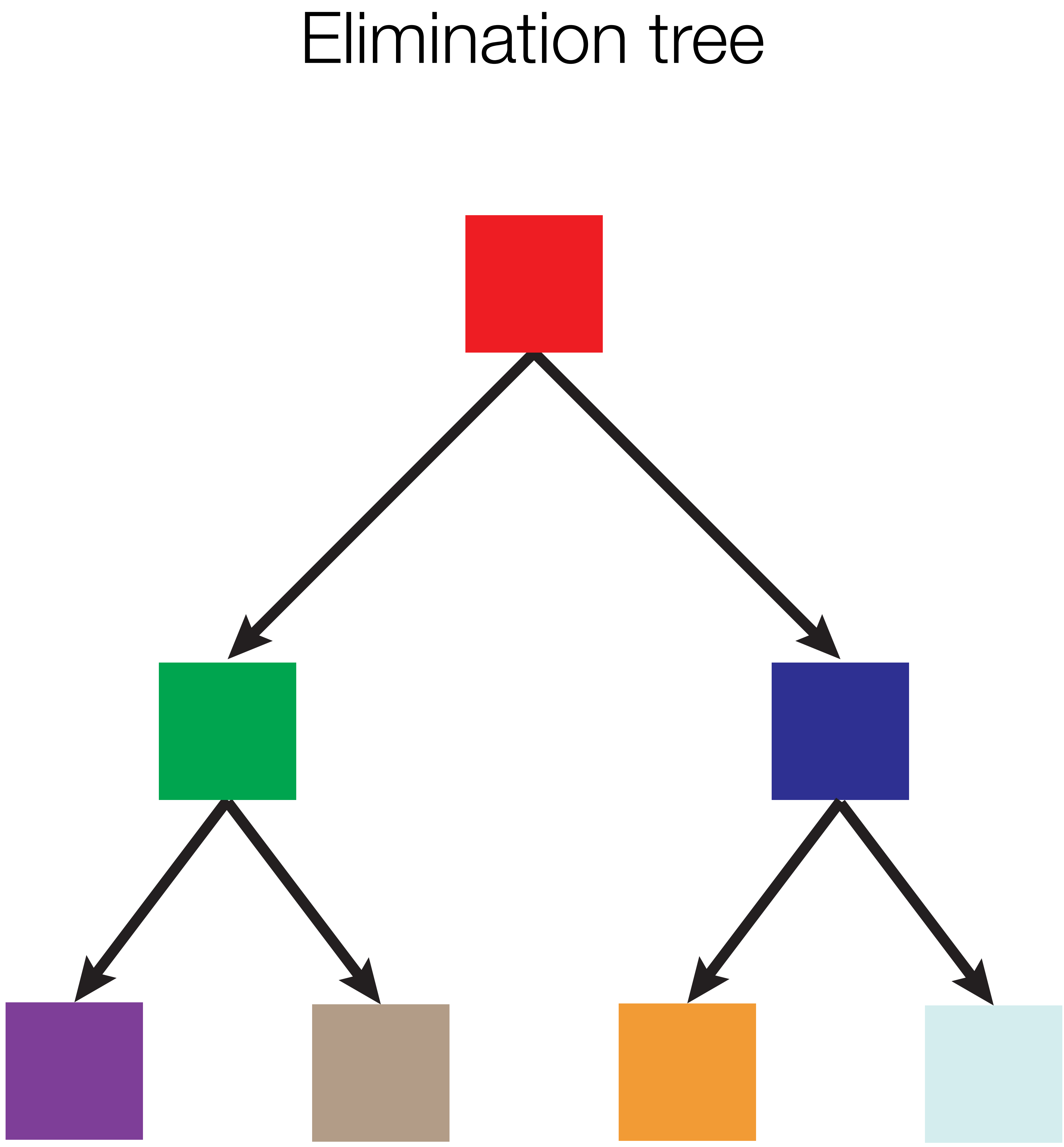


Elimination tree



Nested dissection reordering





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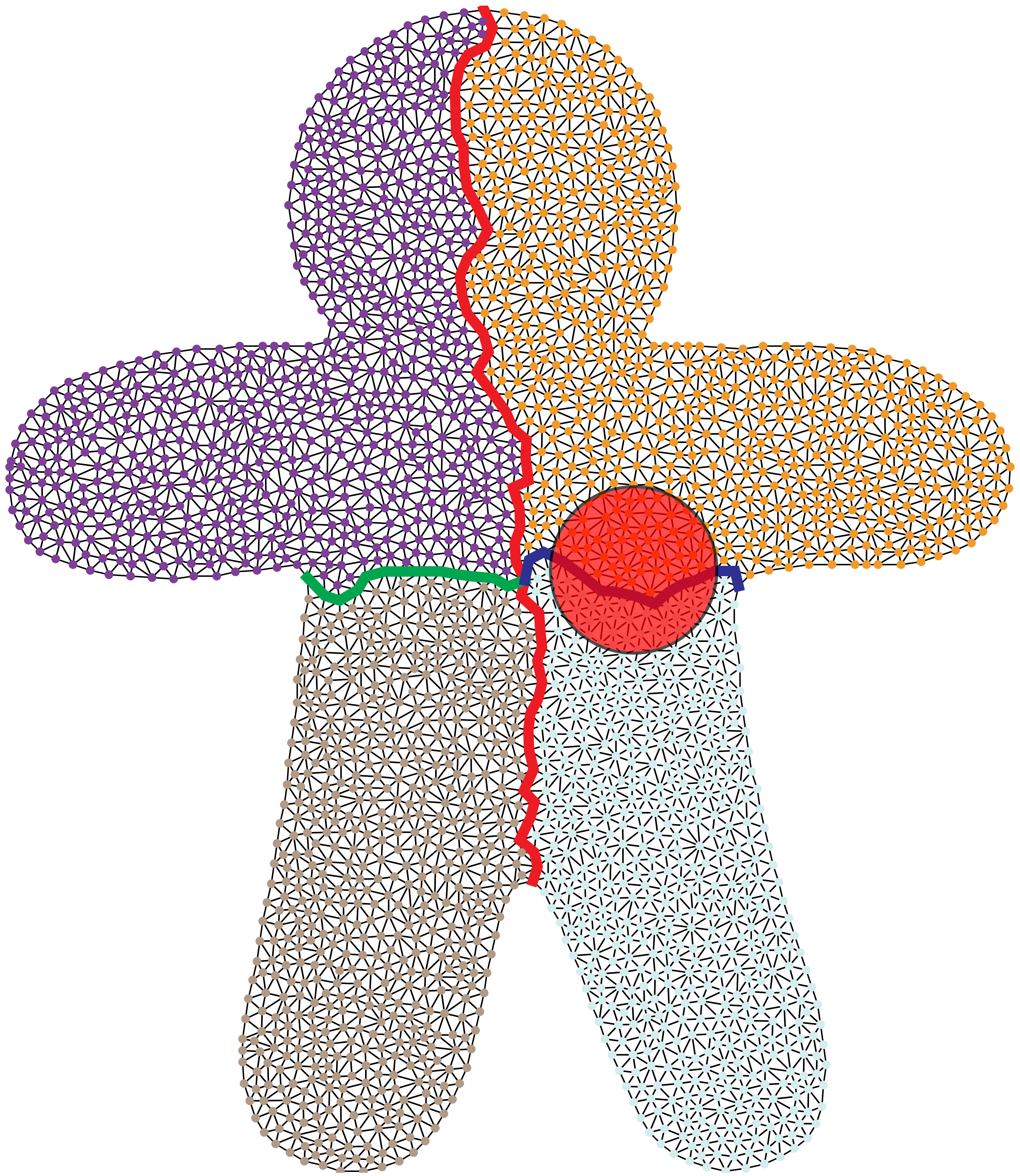

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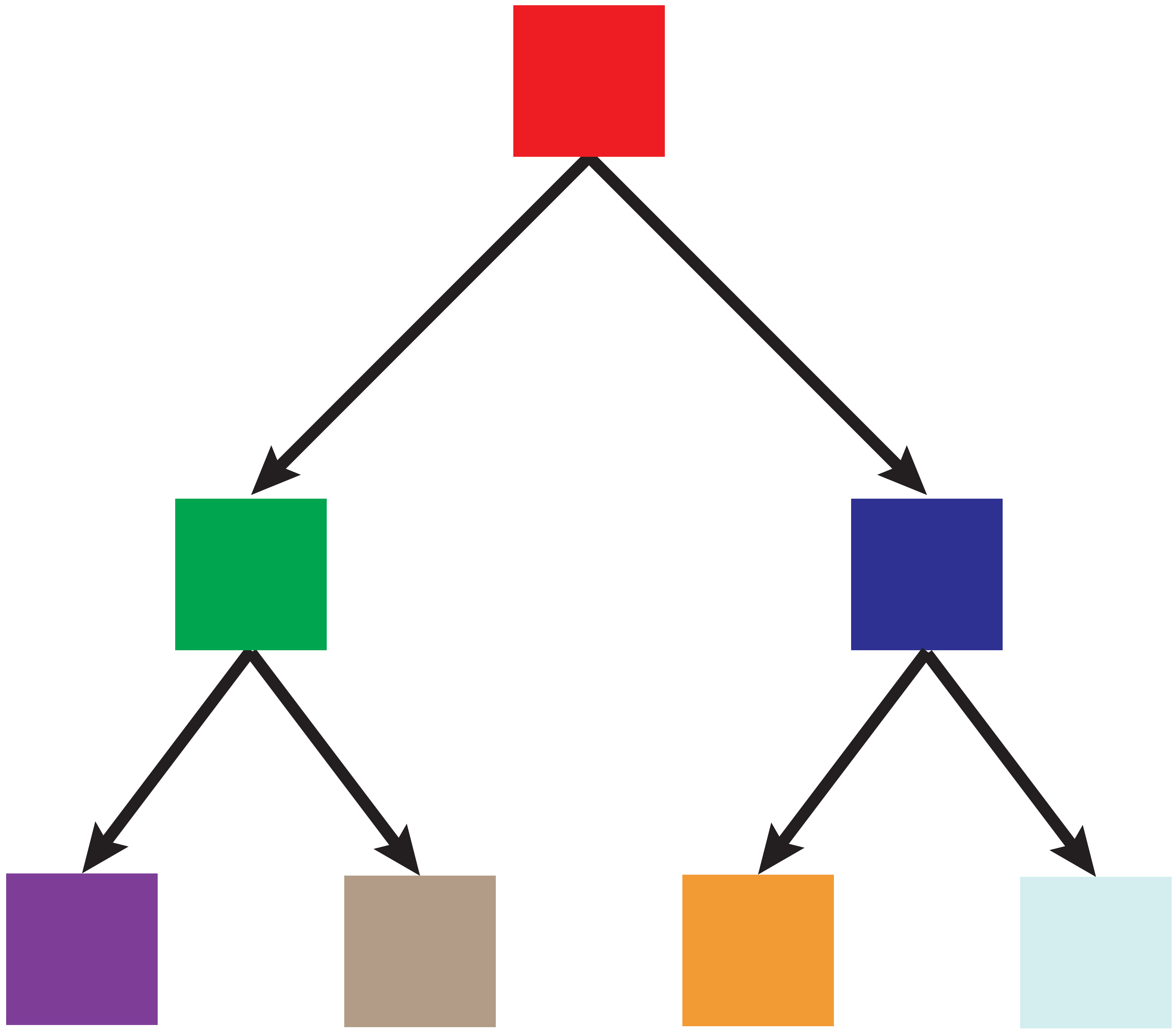

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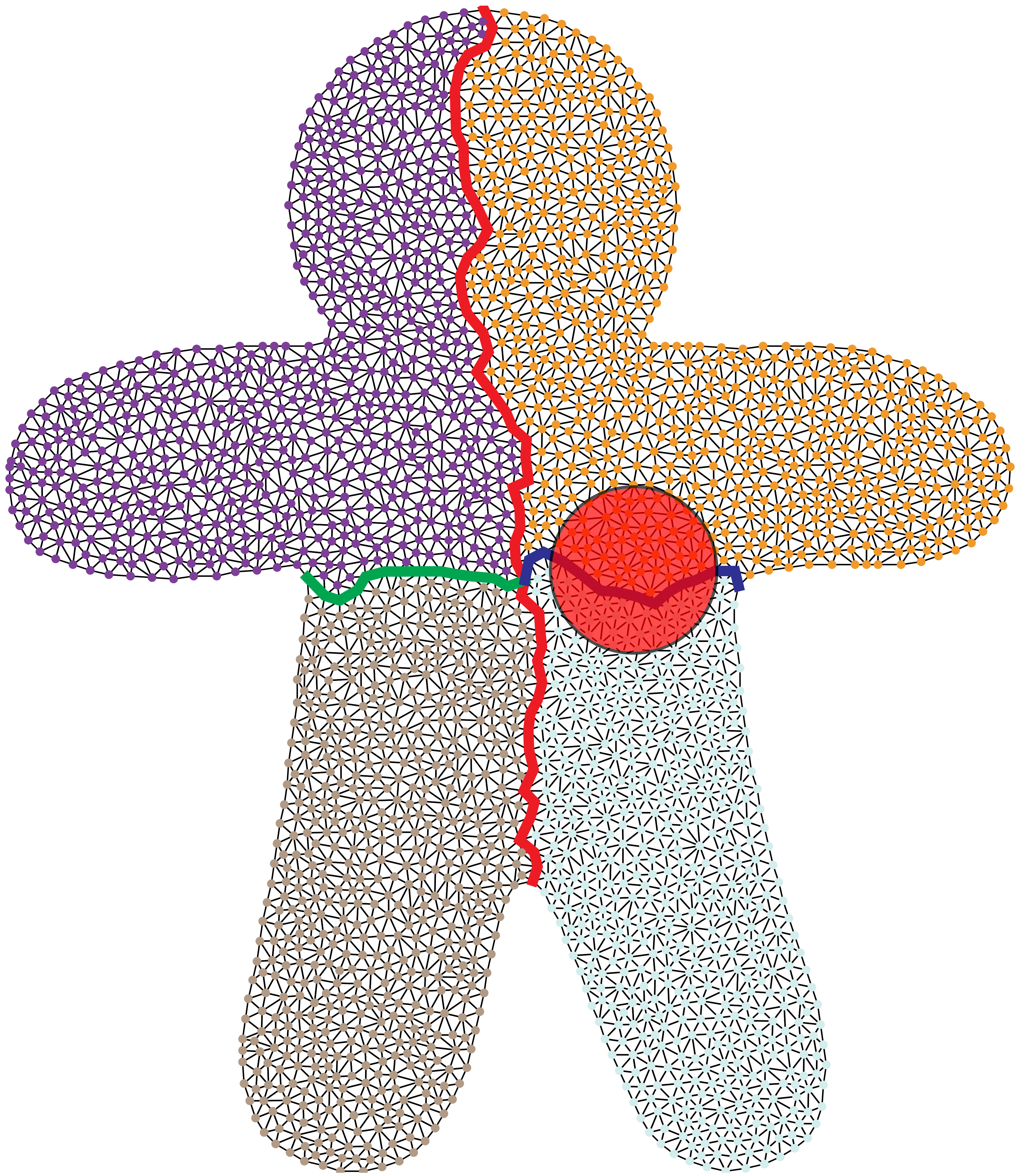

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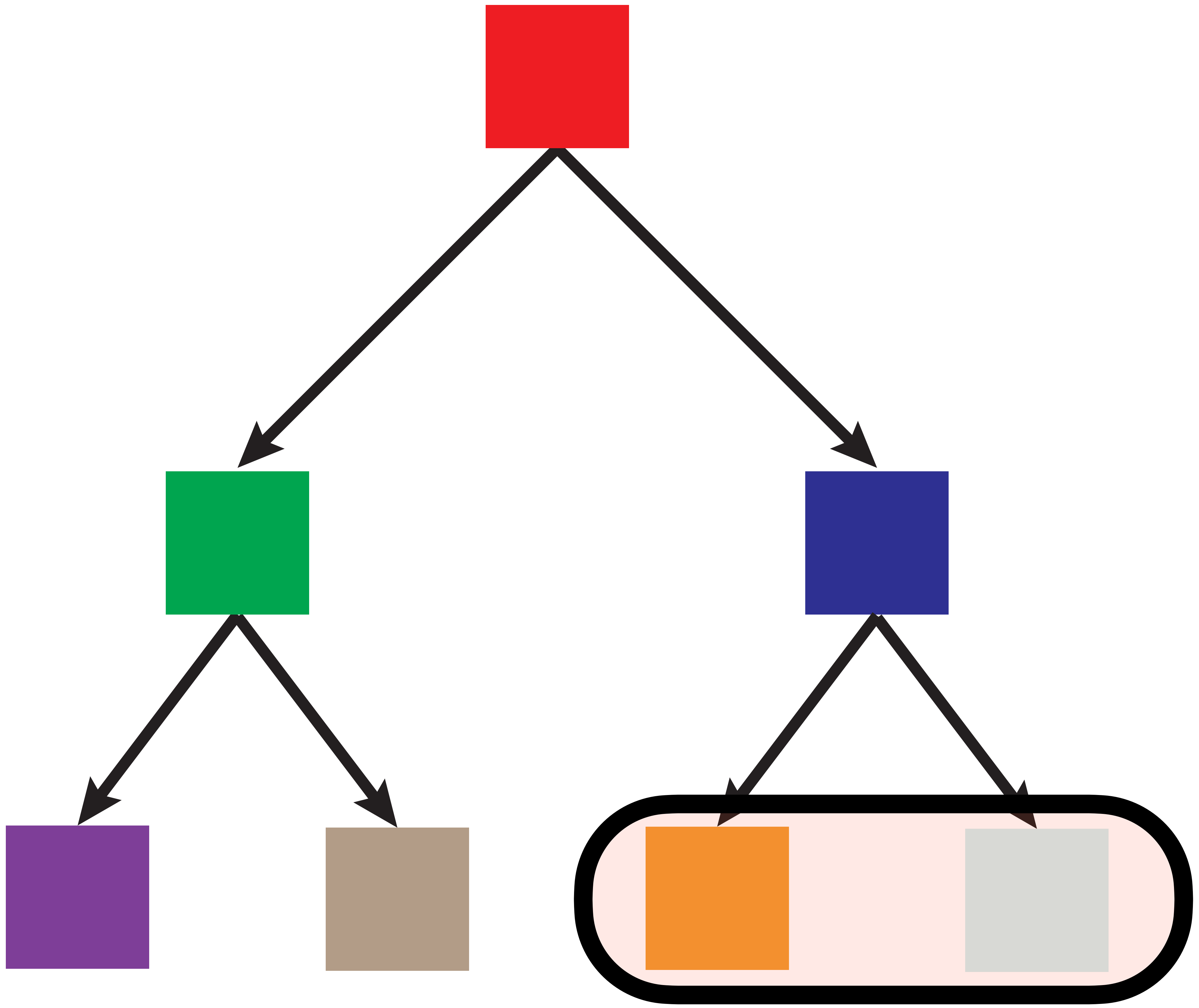
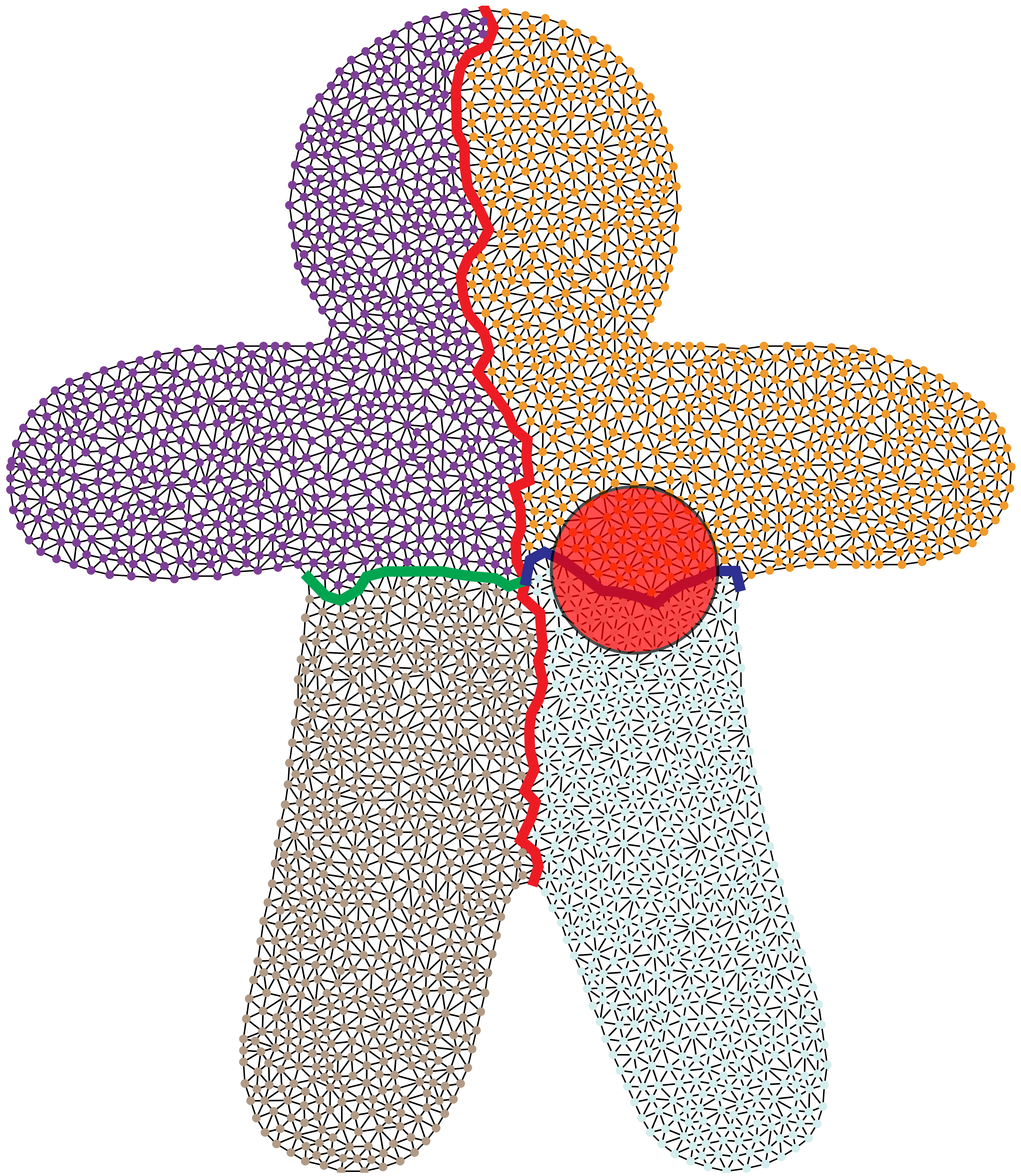
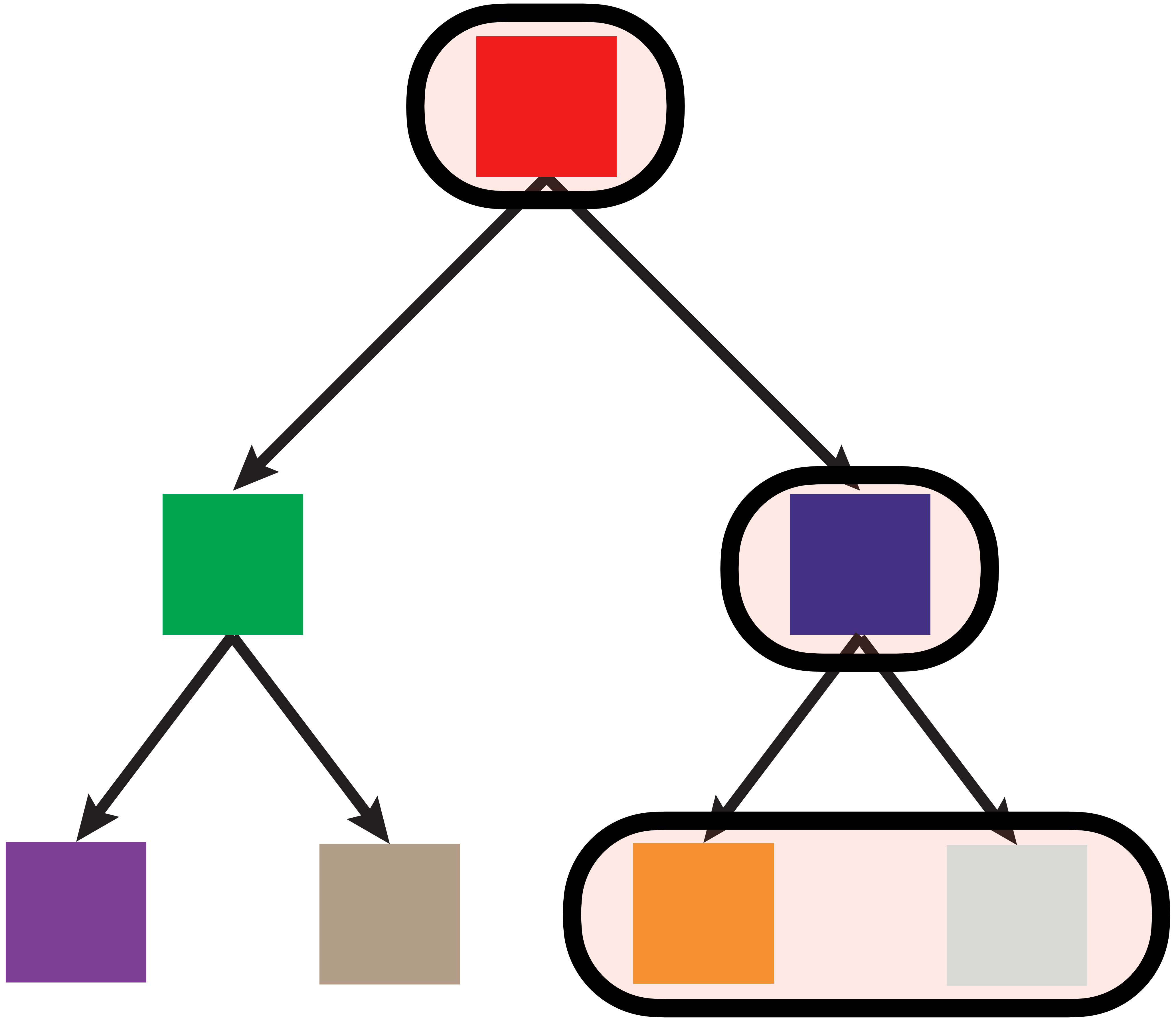


Nested dissection reordering





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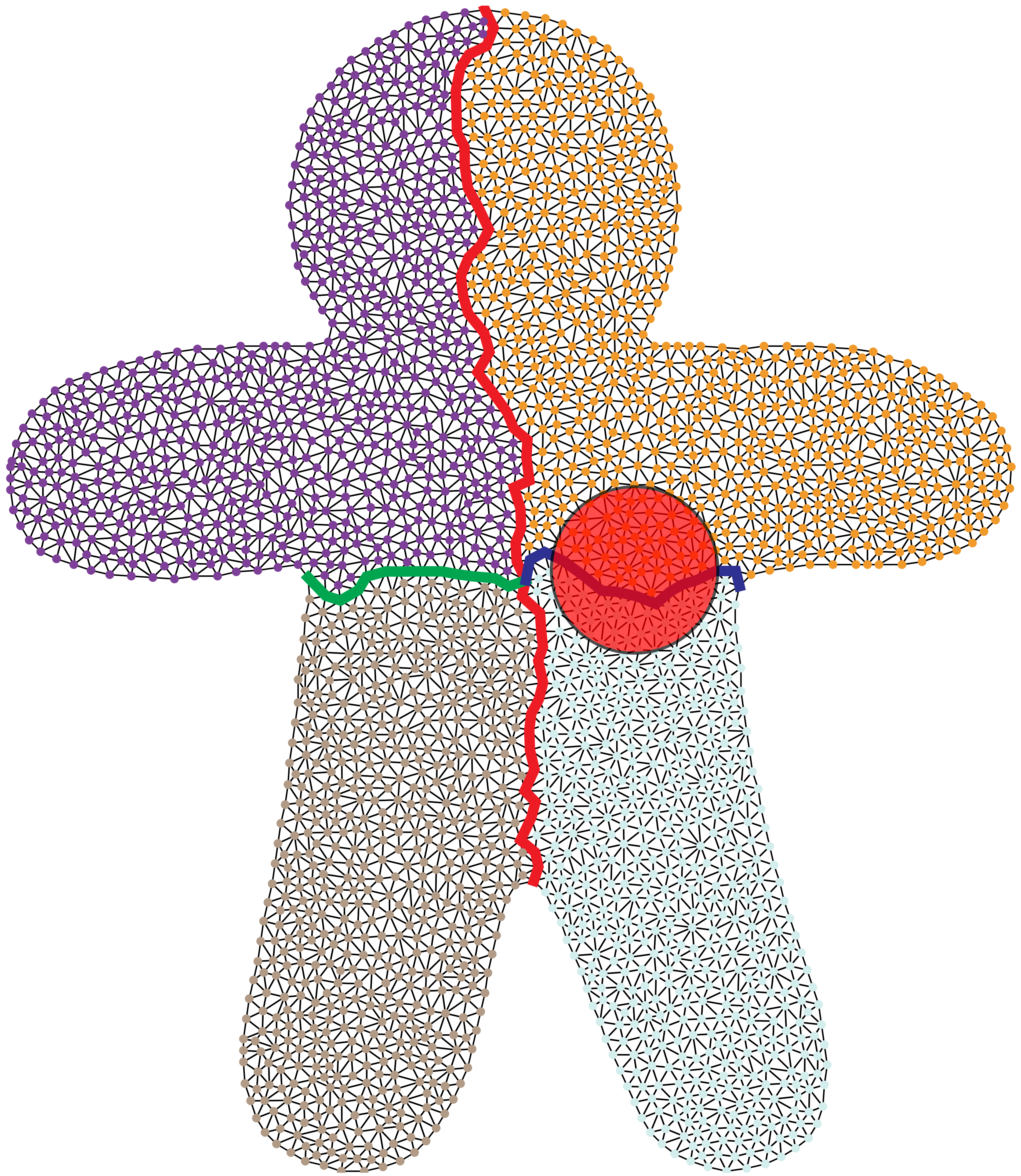

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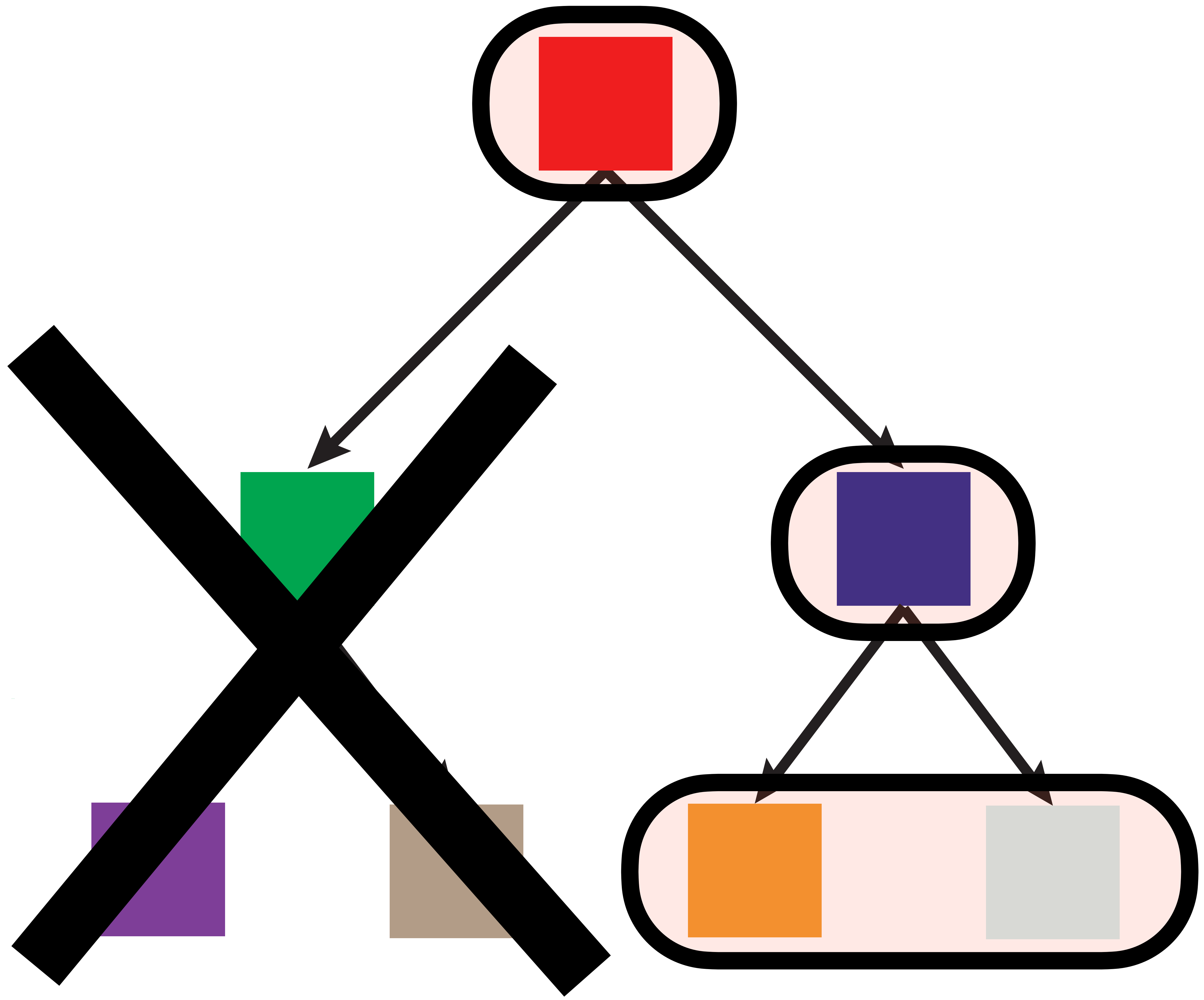

Nested dissection reordering

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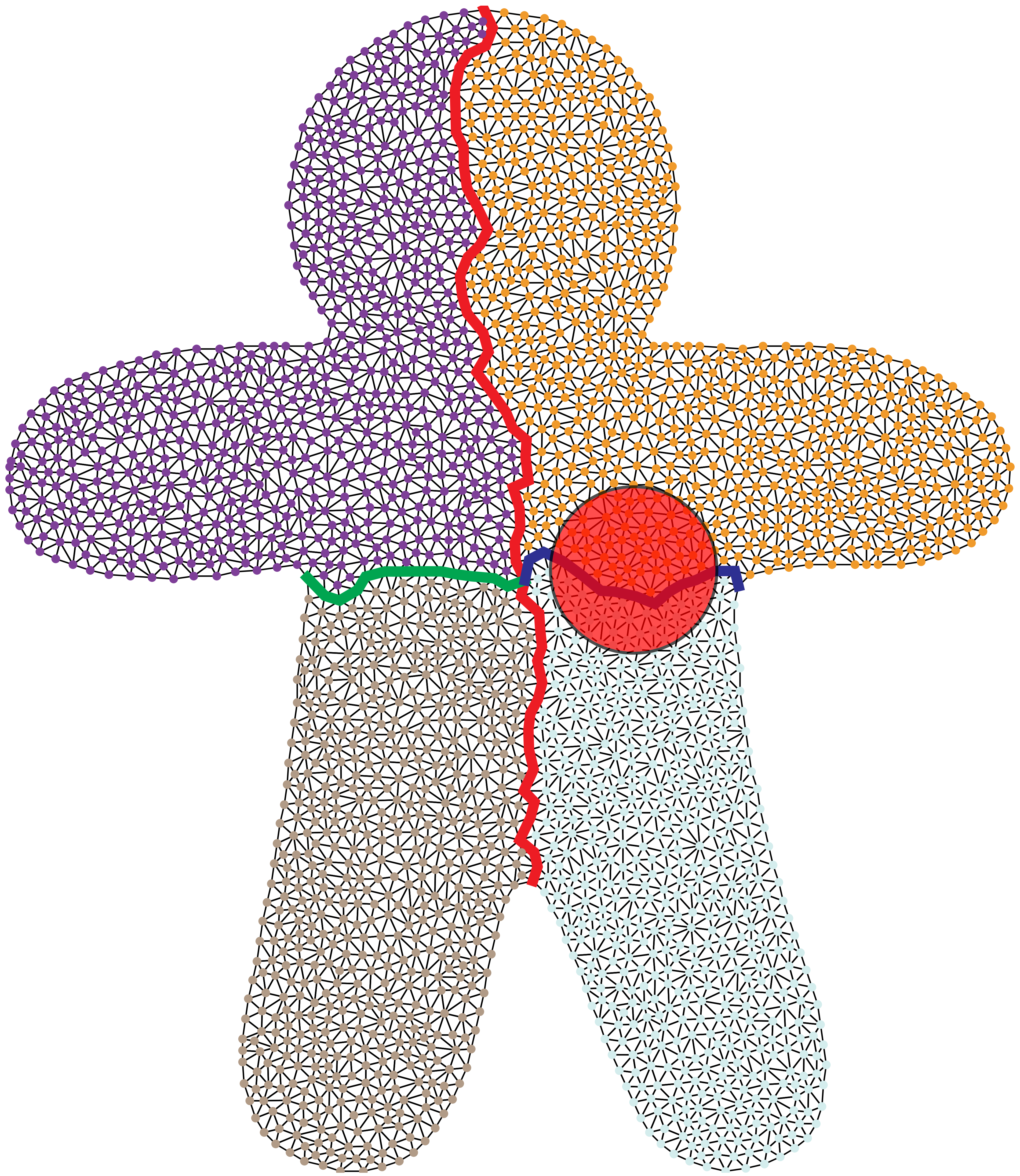
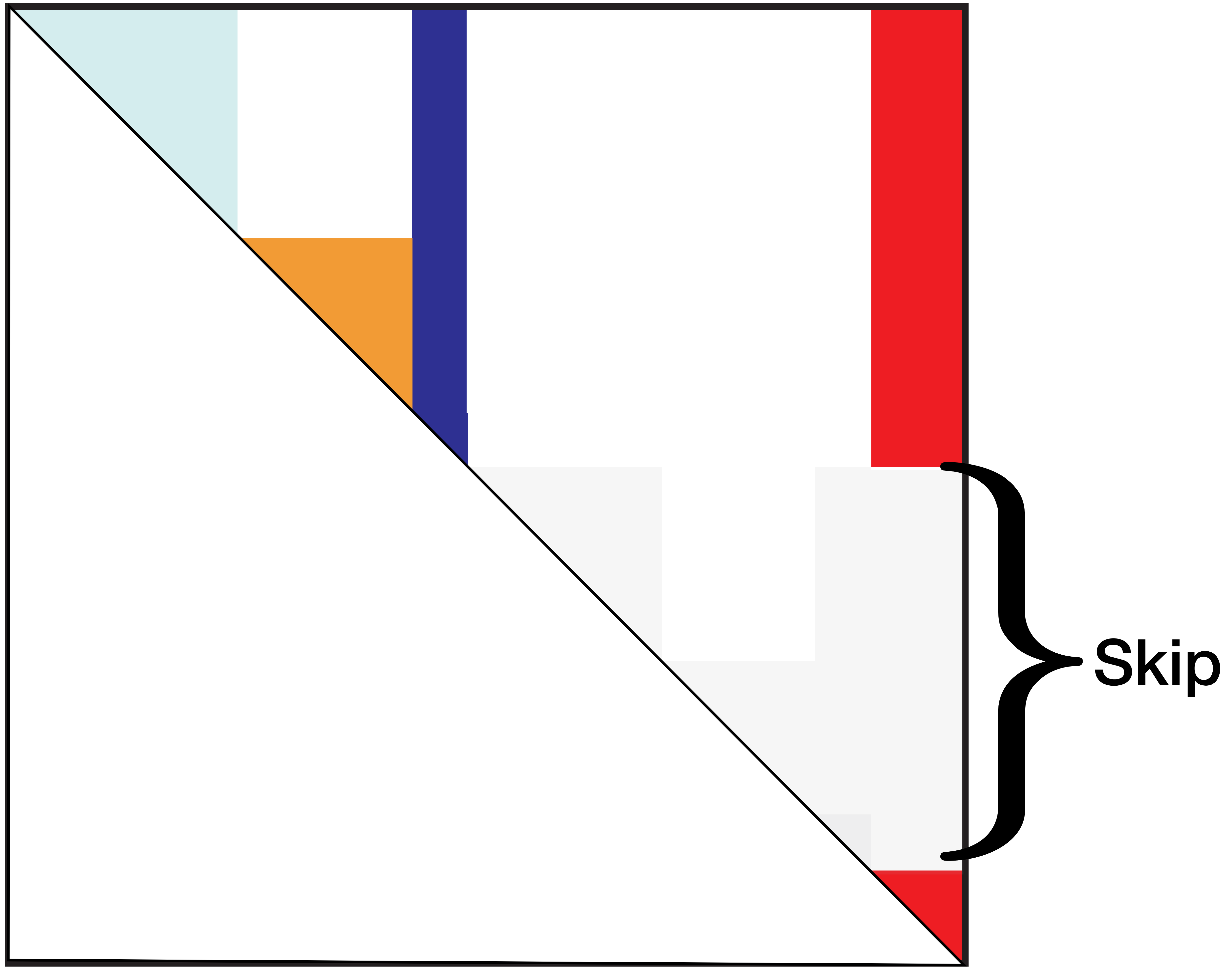
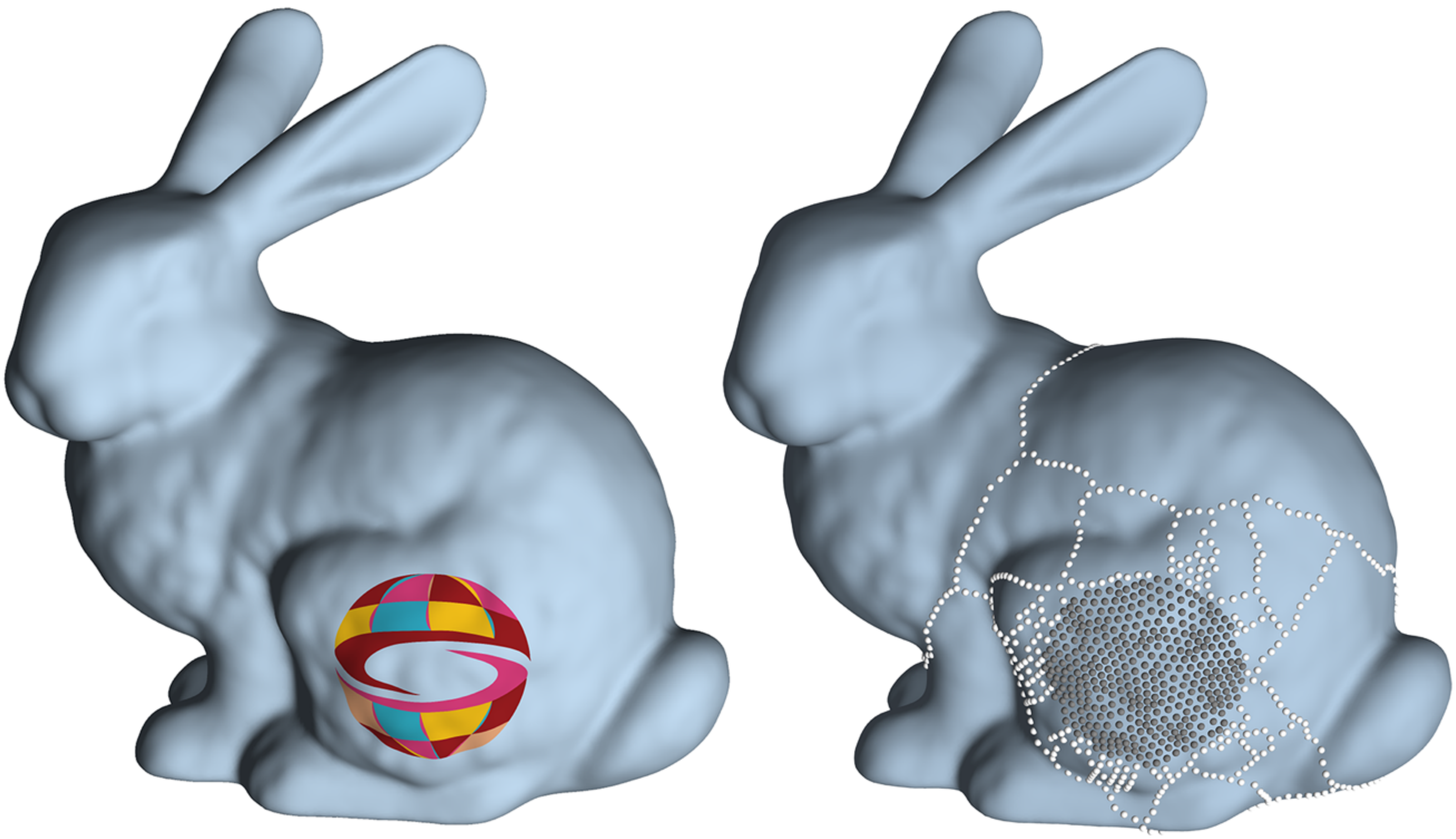


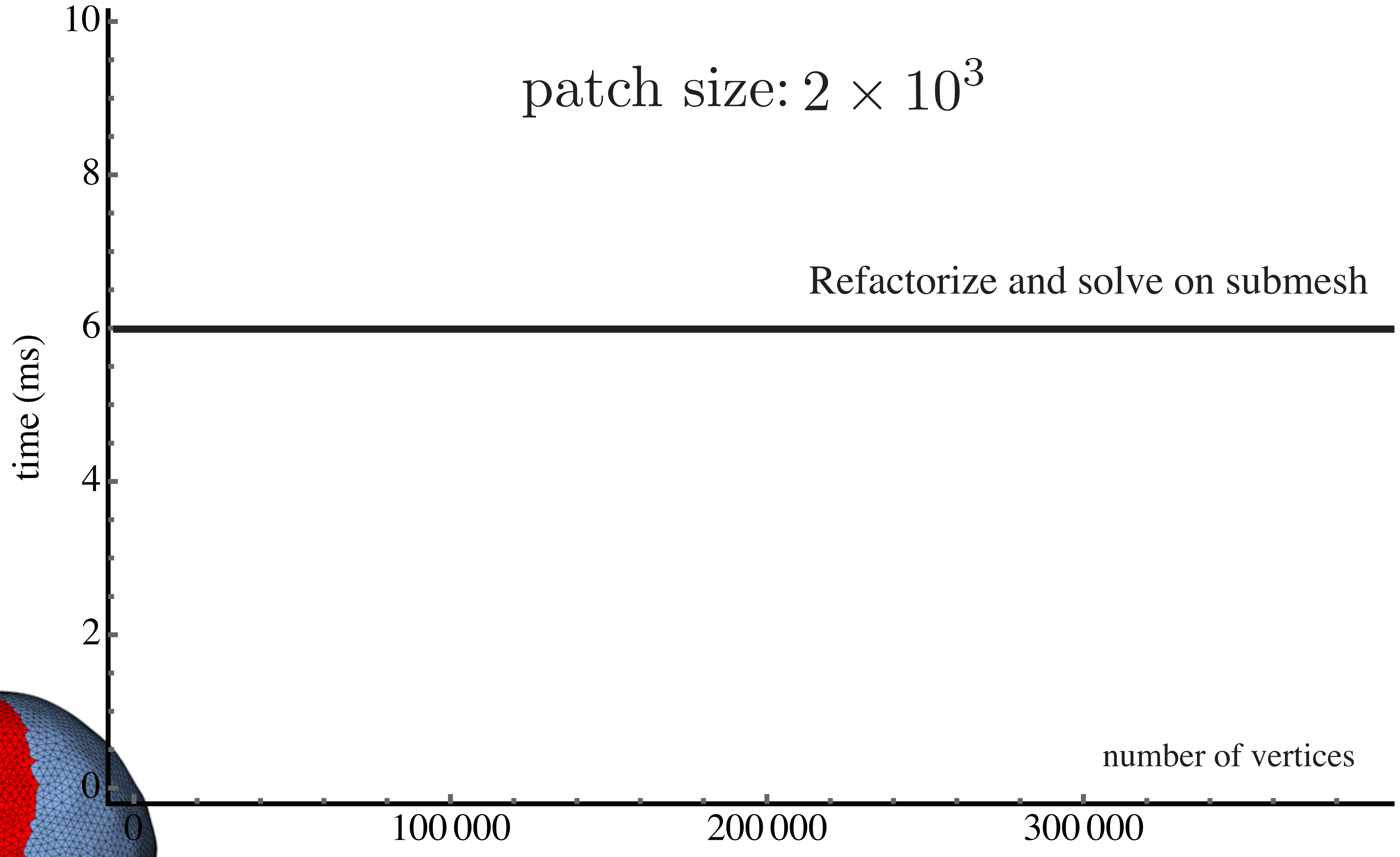
Cholesky factor

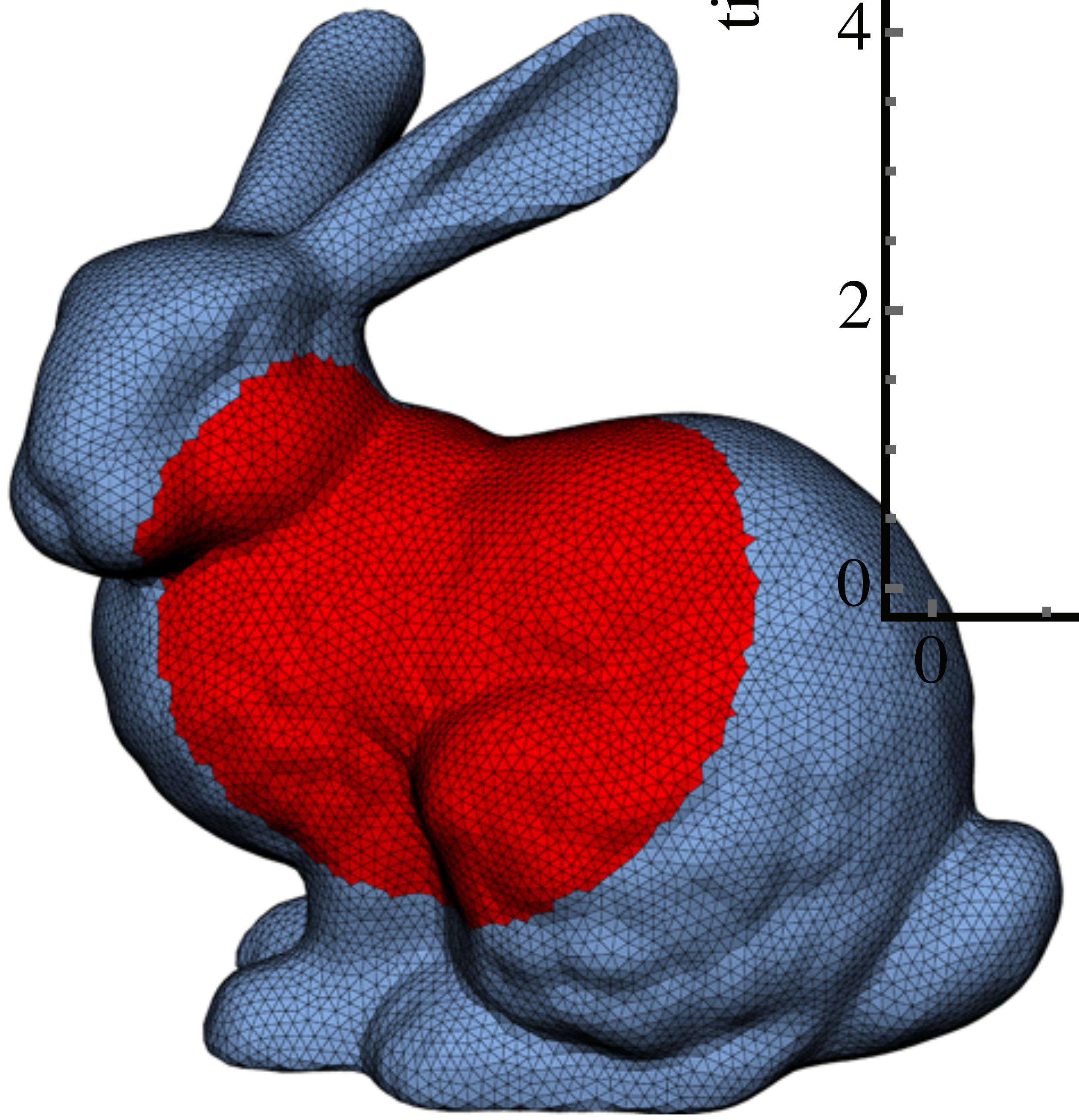


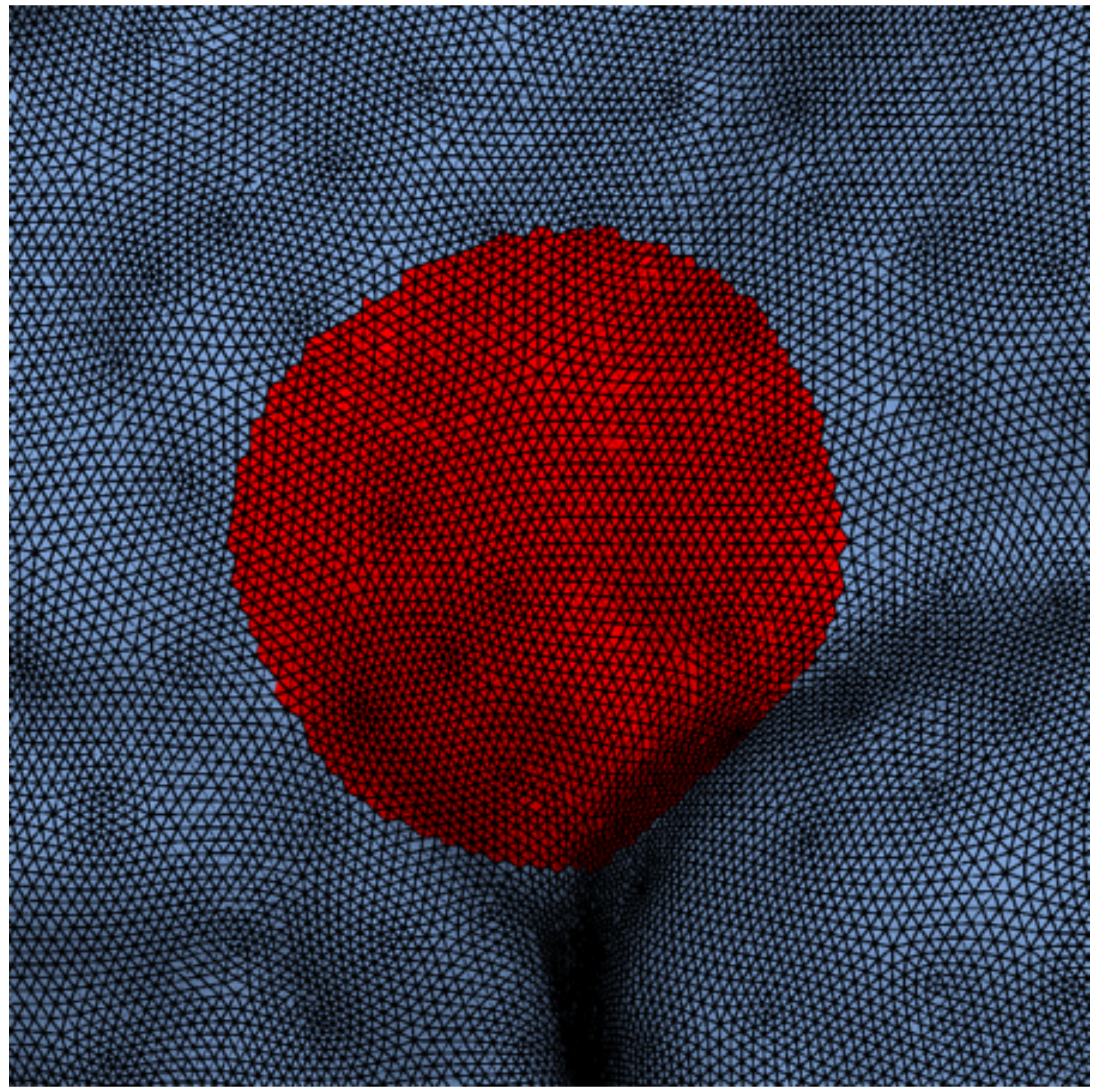
Applications: Parameterization



Is it worth it?







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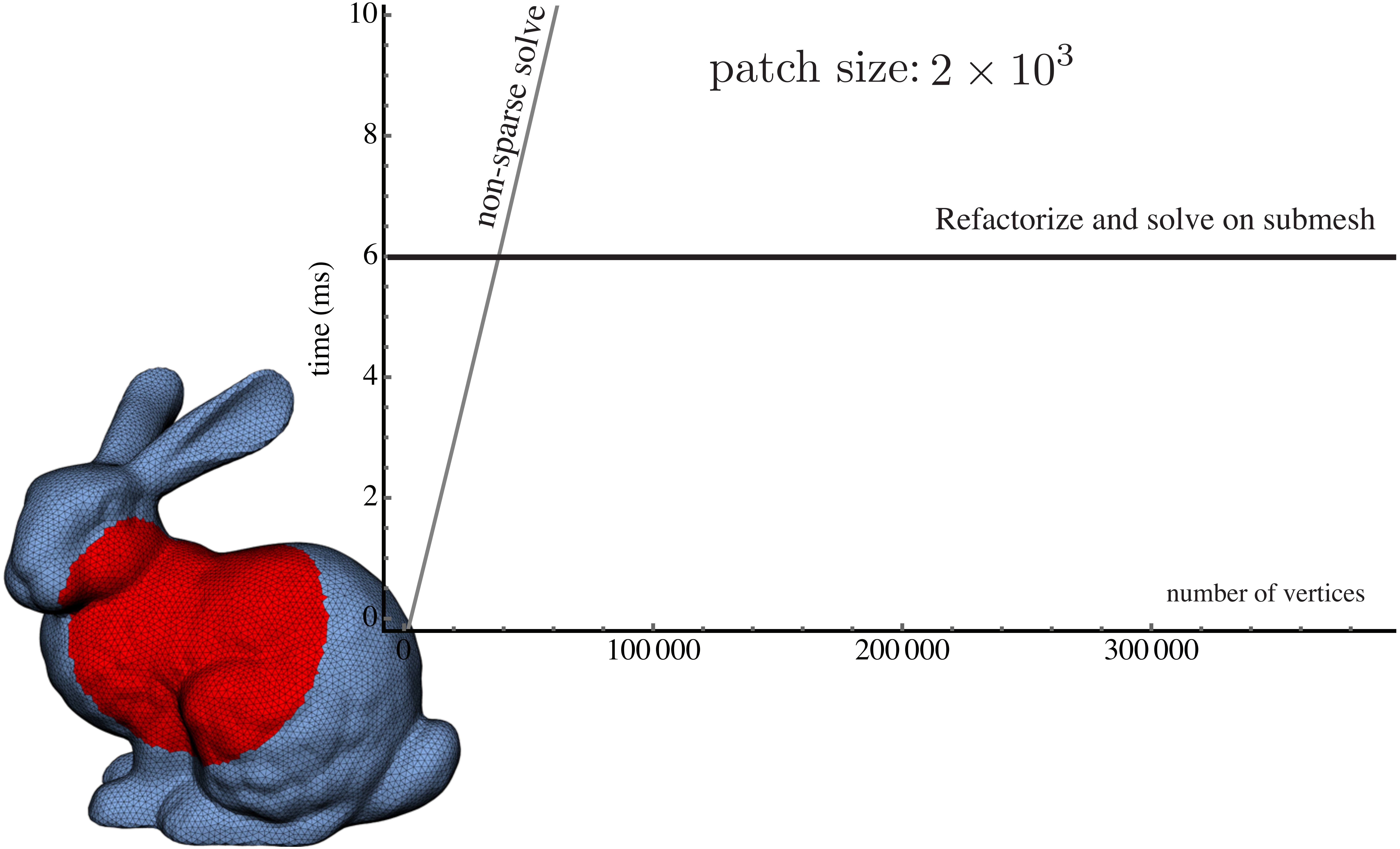

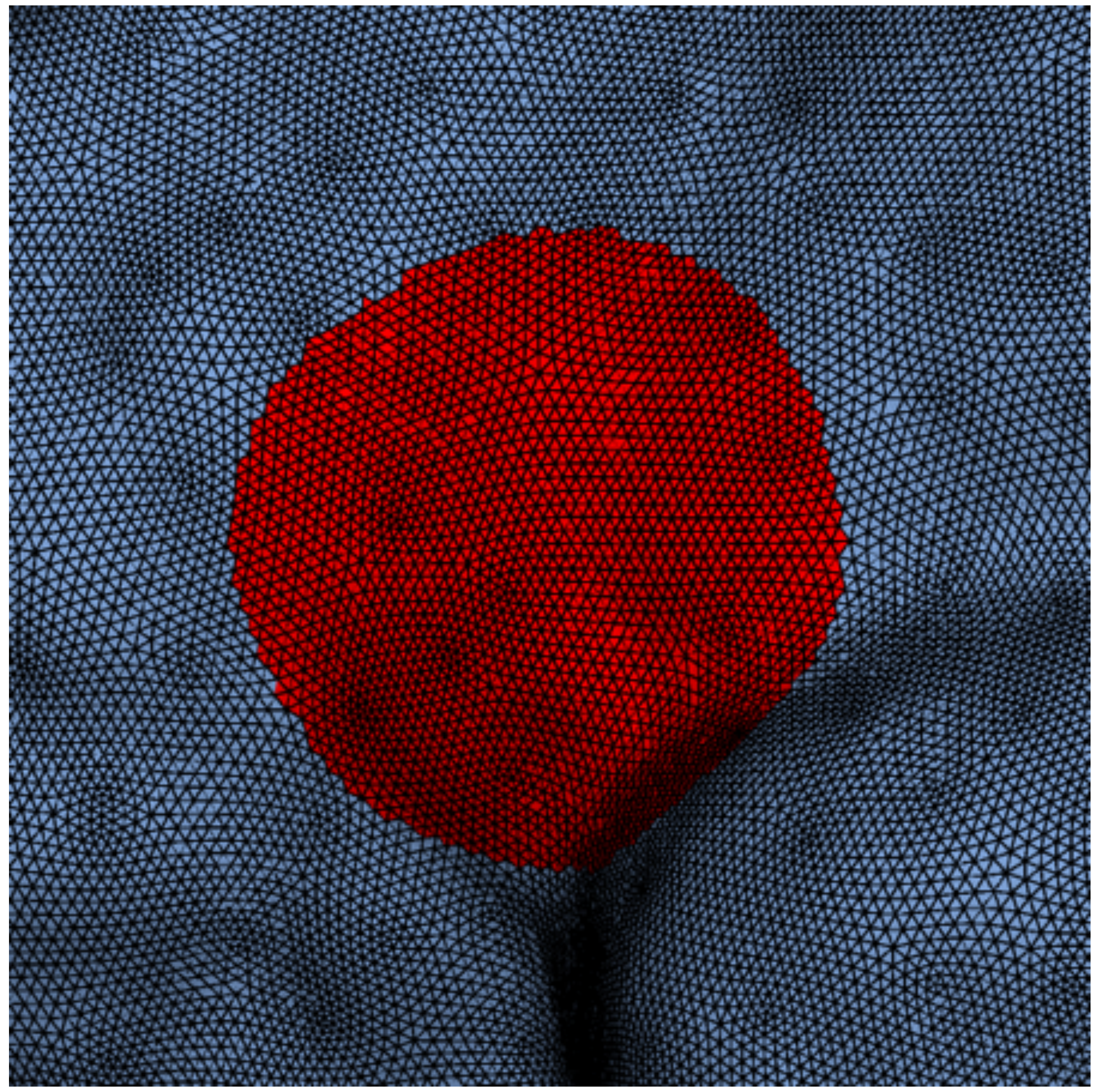

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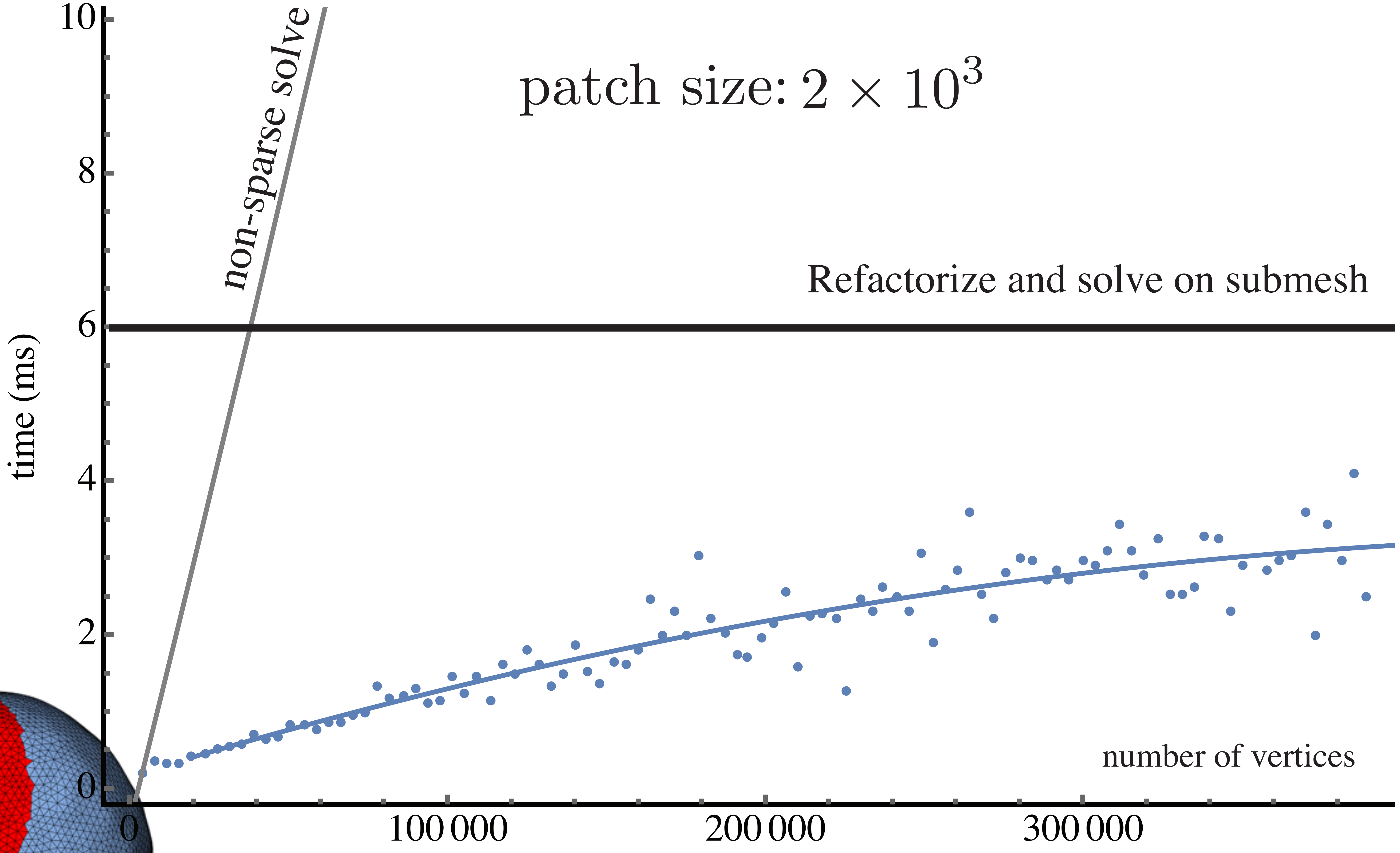

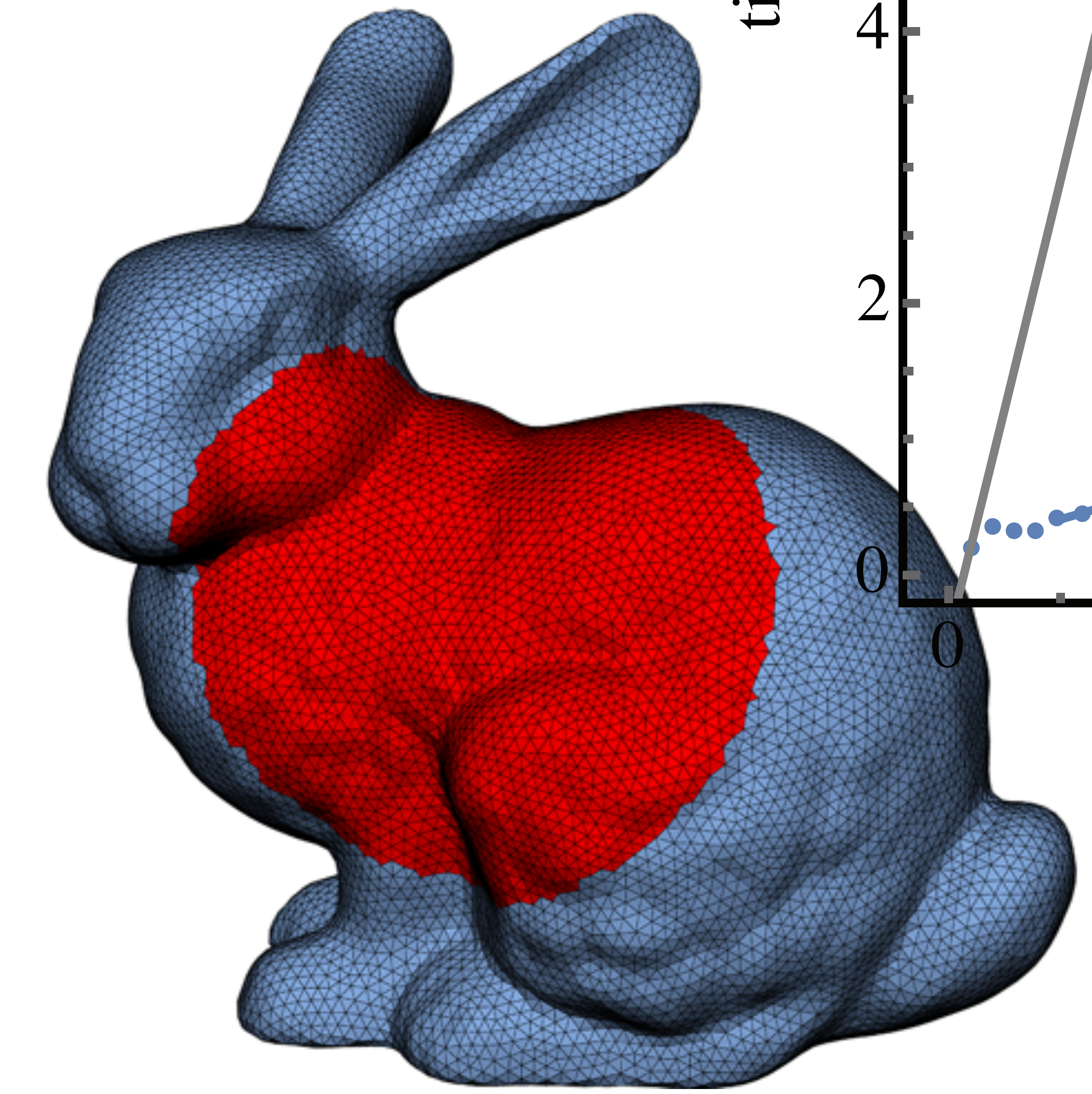

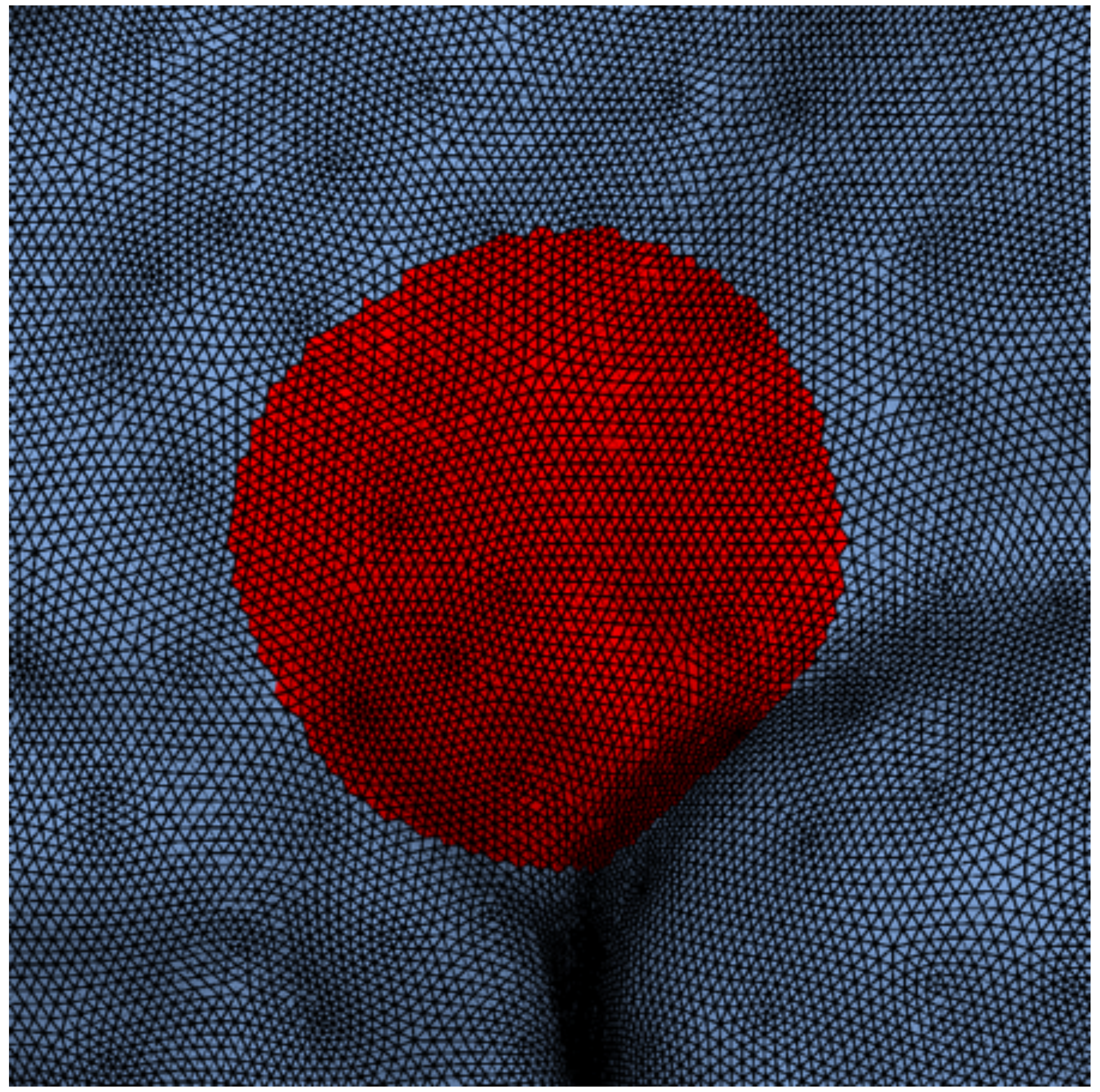

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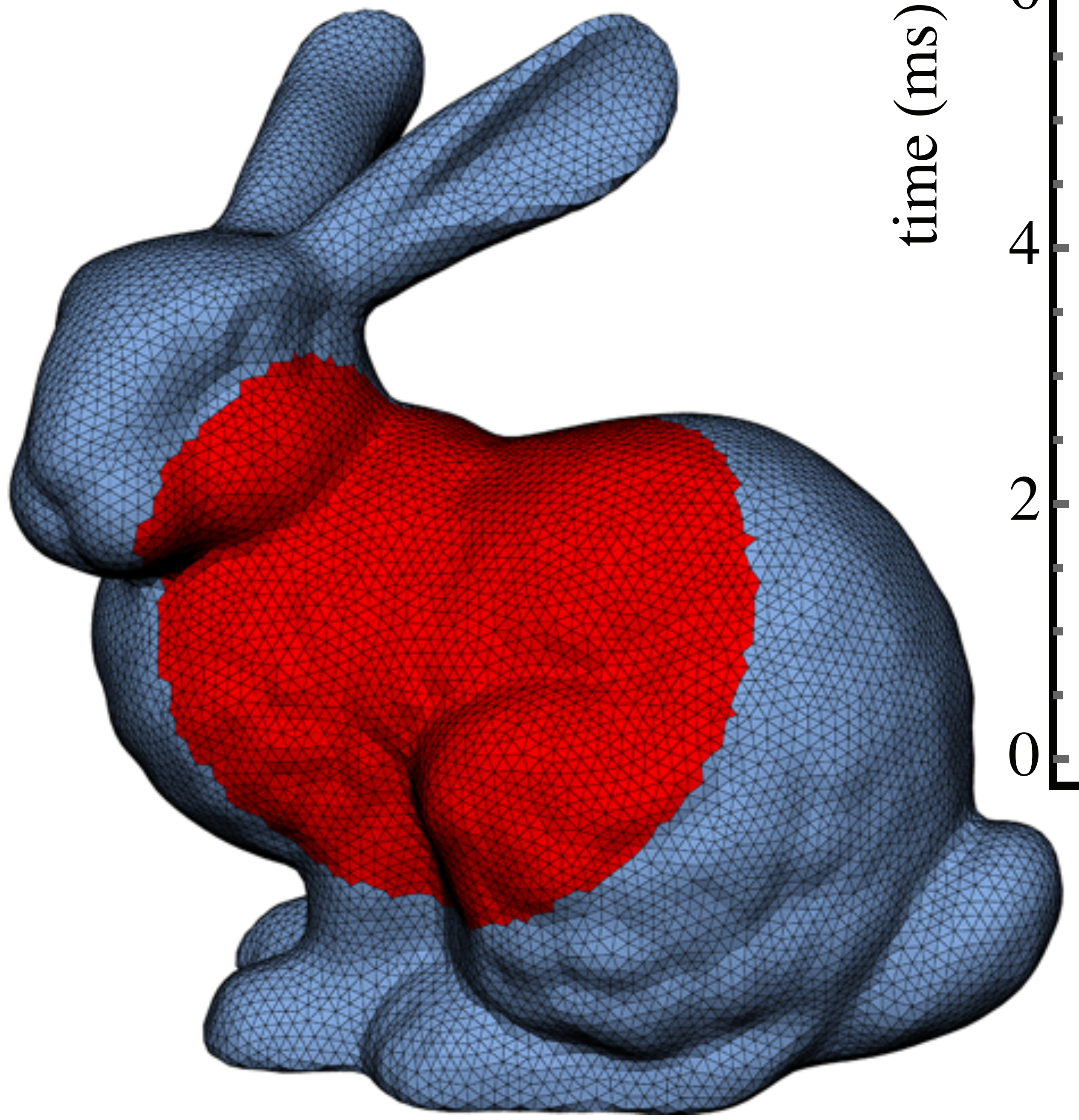

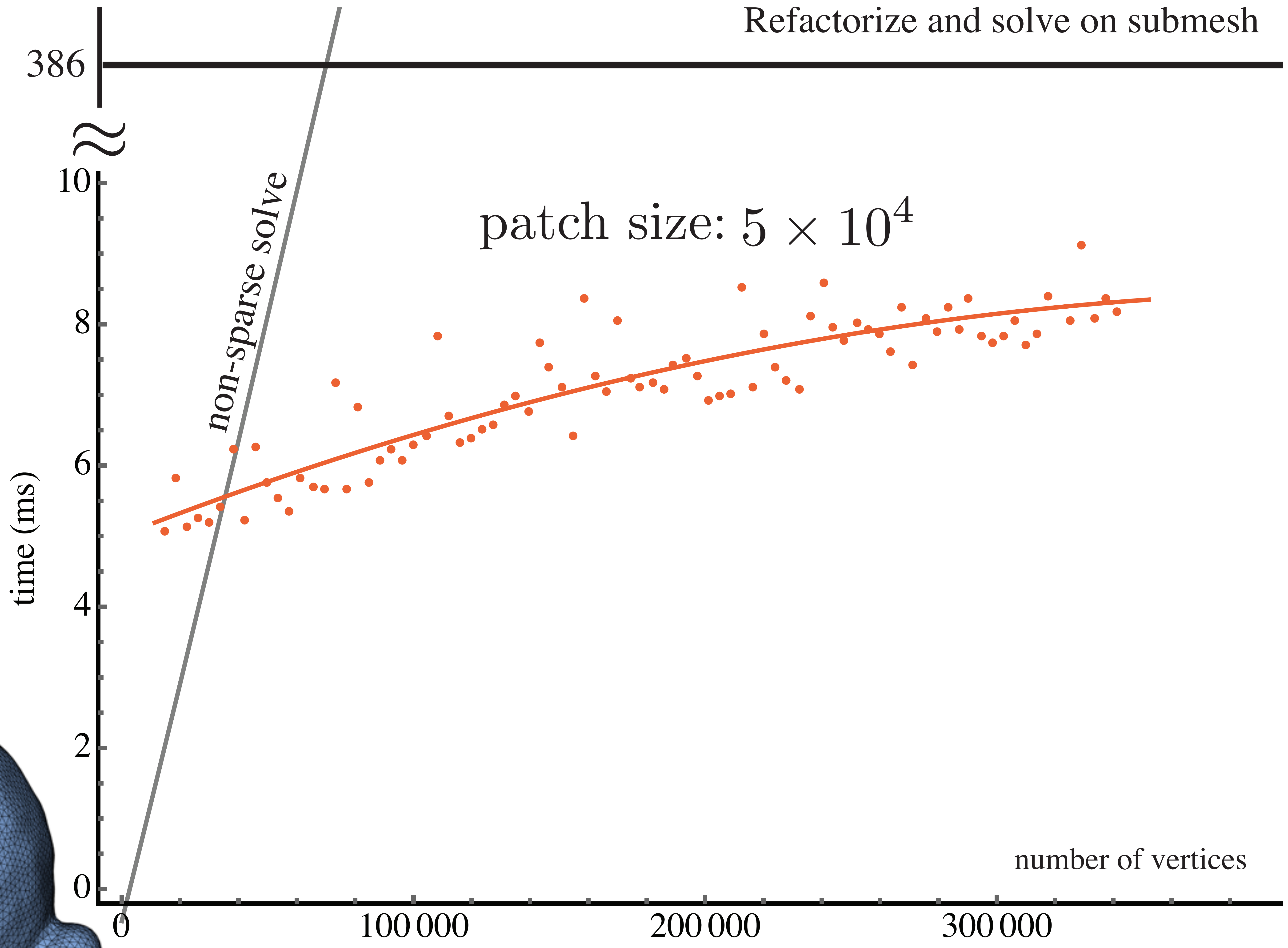

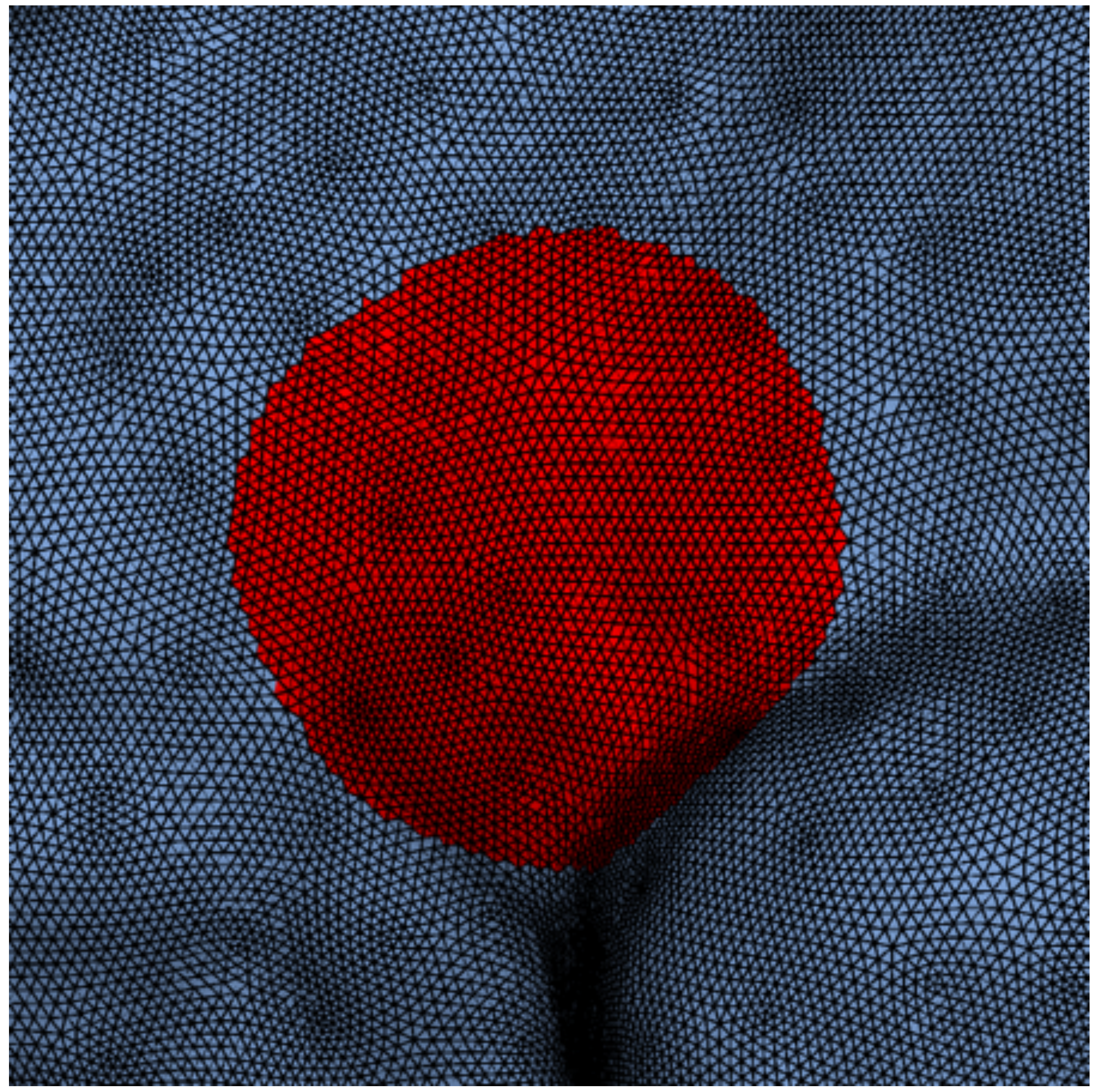

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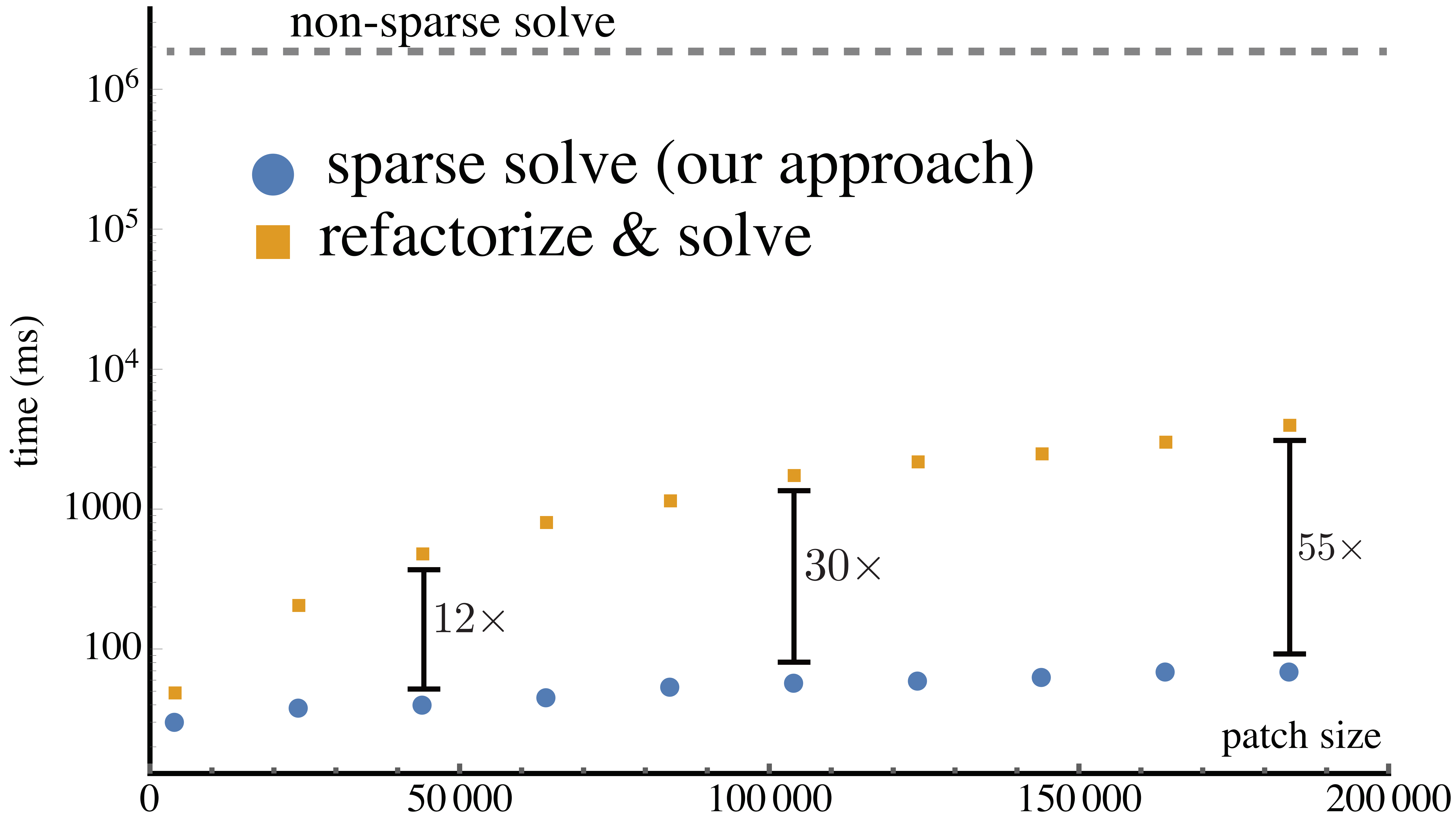
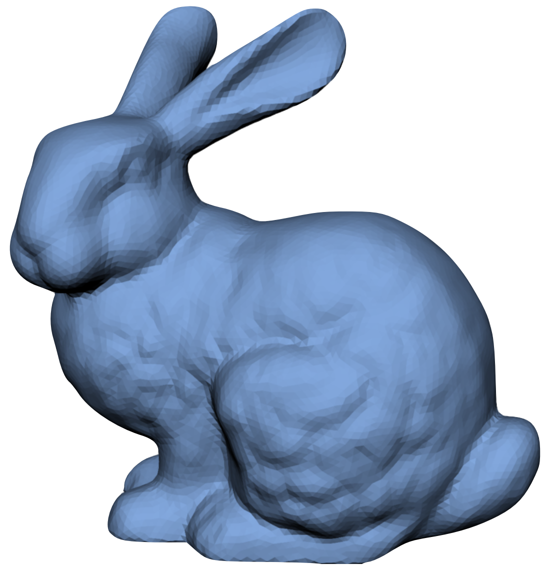


Amortization of factorization time

$n \times \Big($ $\mathbf{A}_1 = \mathbf{A}\big($  $\big)$

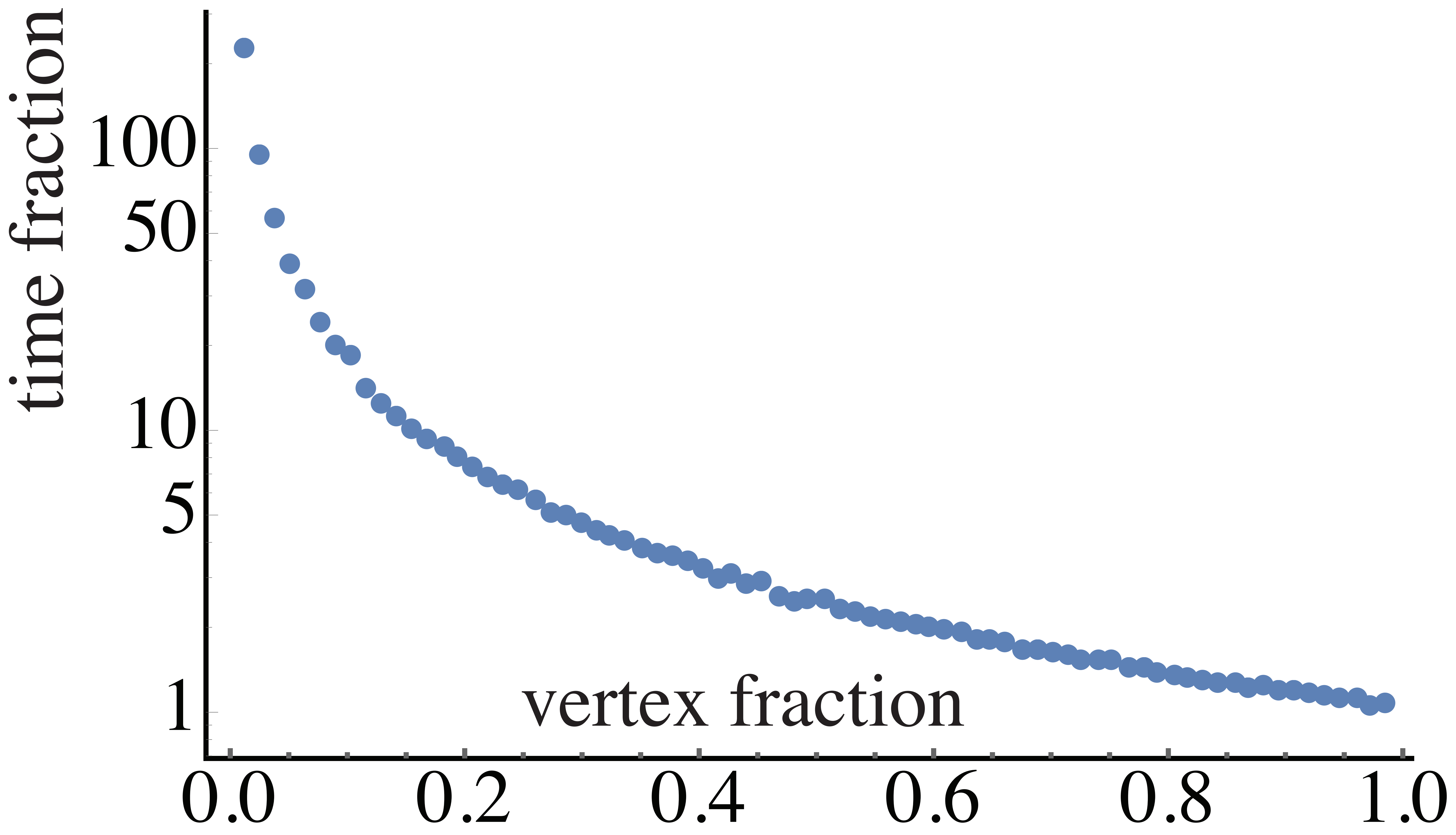
$\mathbf{A}_1 = \mathbf{LDL}^T$ $\Big)$

vs

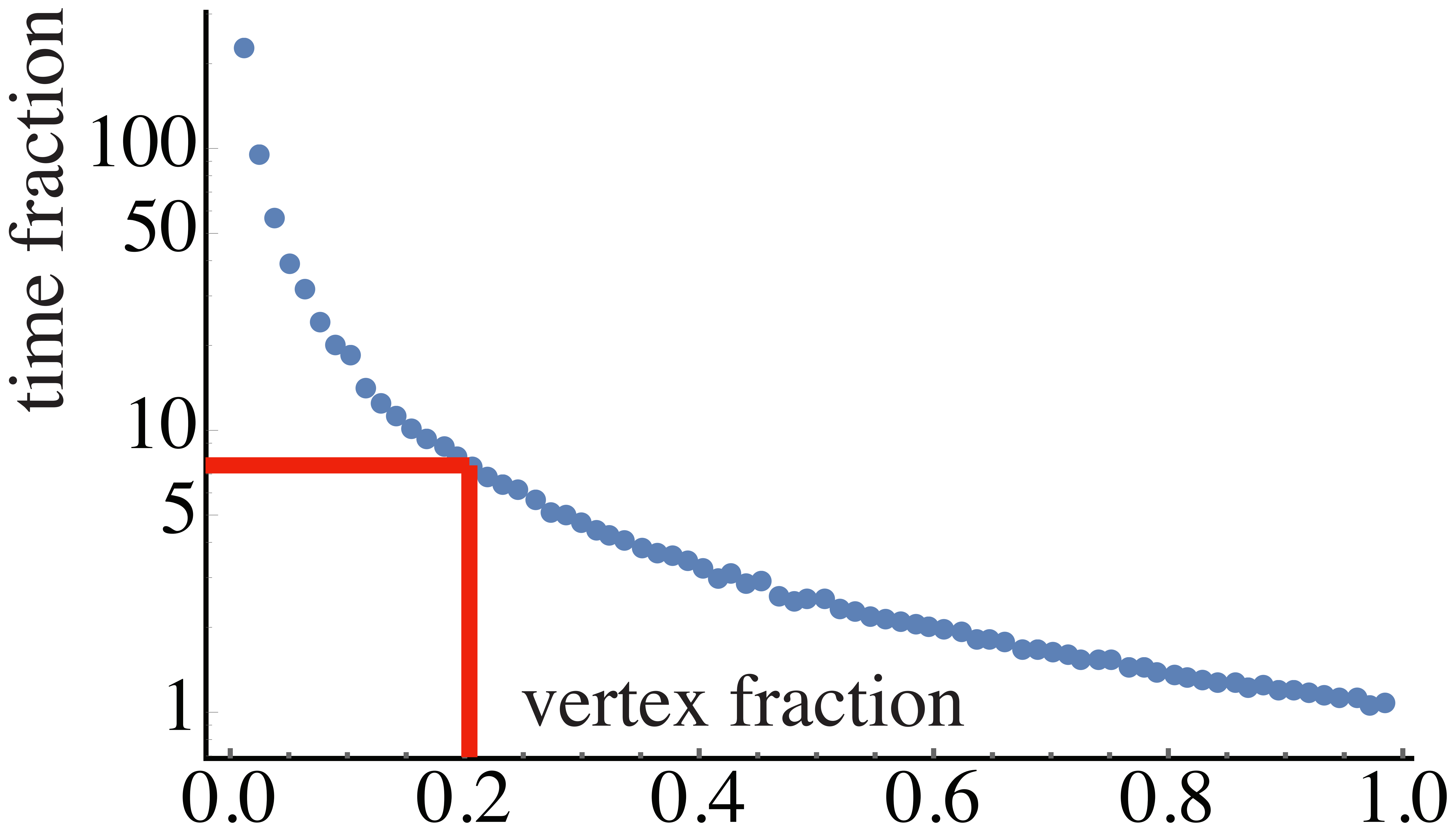
$\mathbf{A} = \mathbf{A}\big($  $\big)$

$\mathbf{A} = \mathbf{LDL}^T$

Amortization of factorization time



Amortization of factorization time



Limitations

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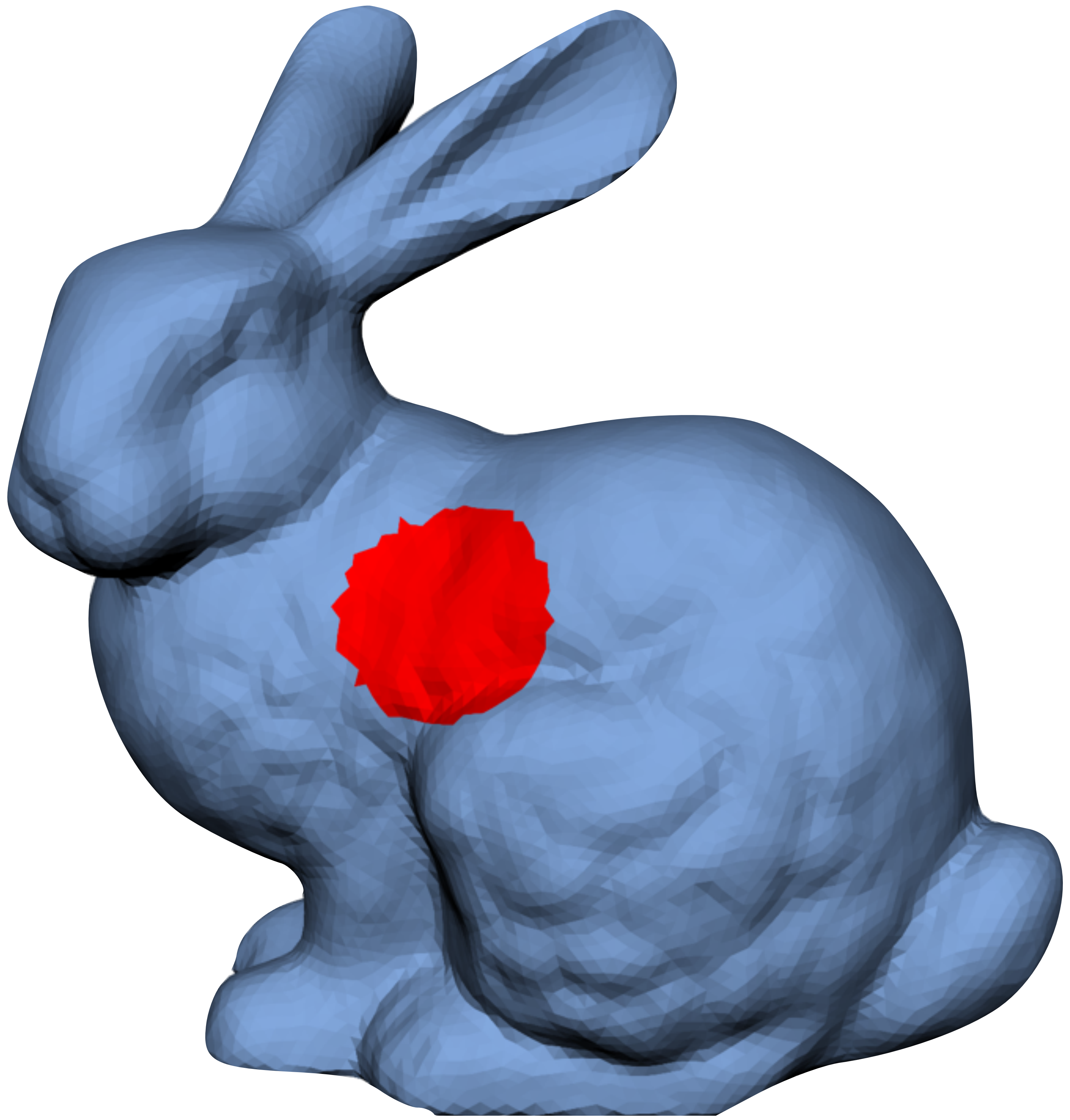

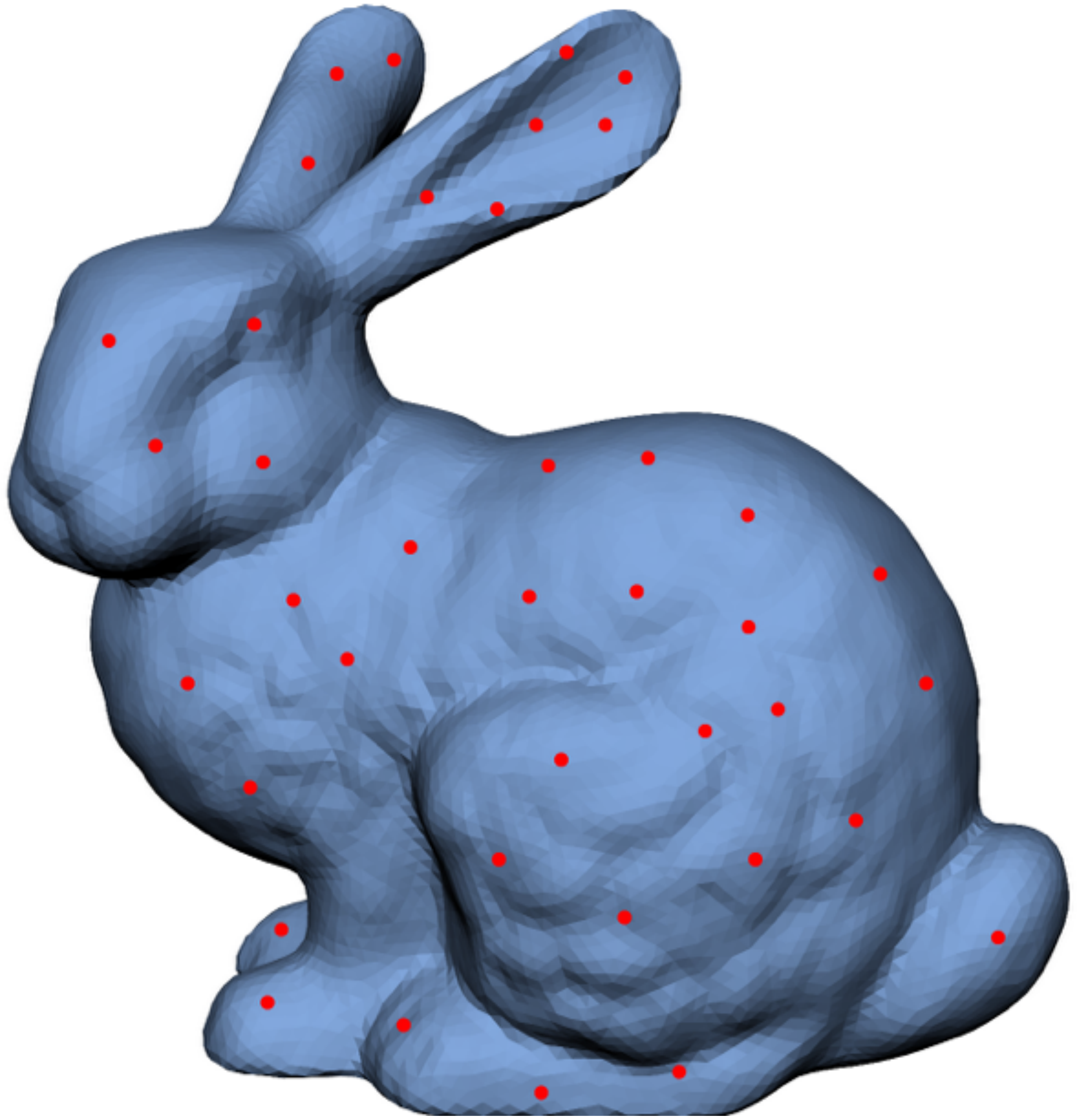

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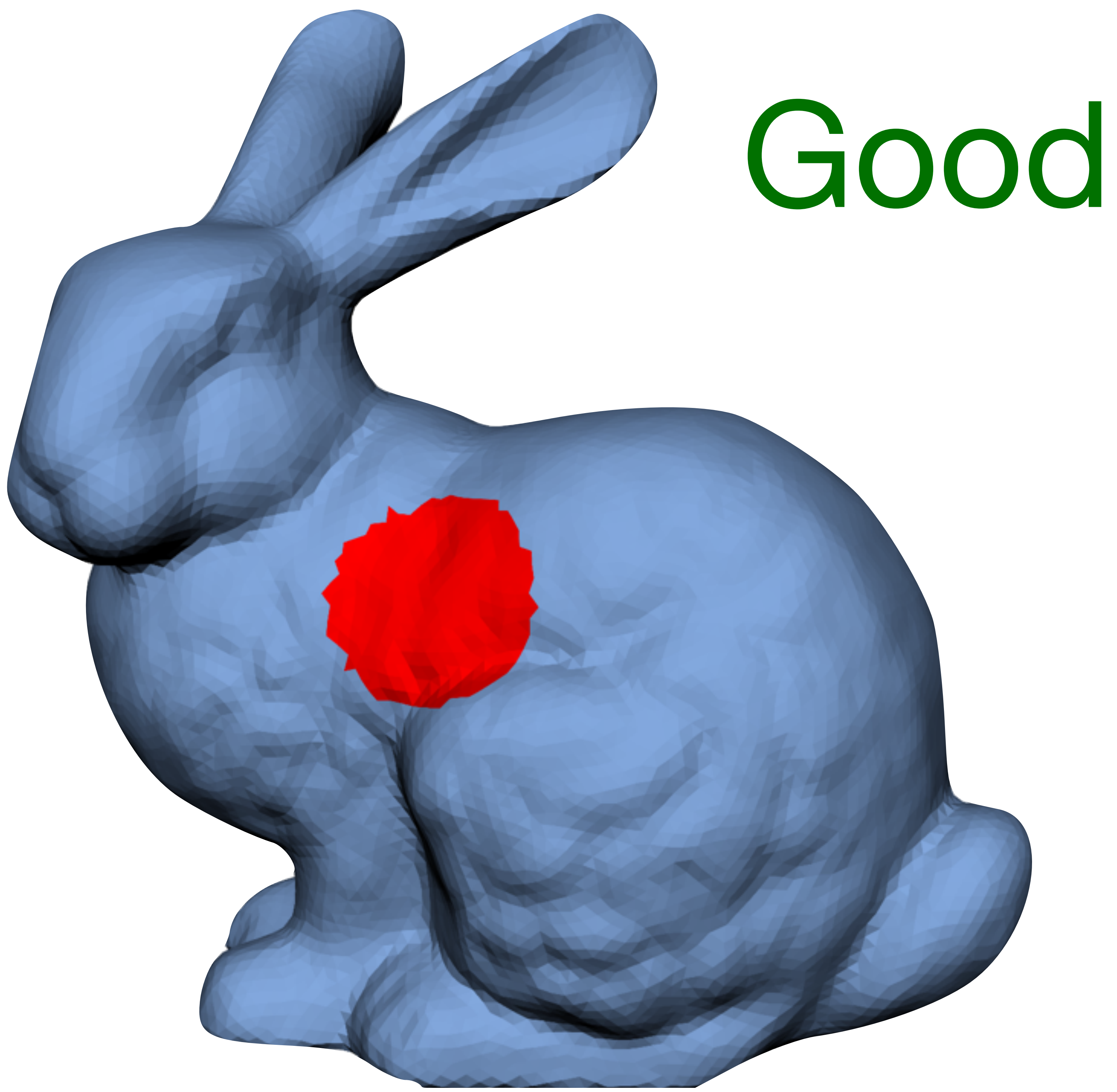

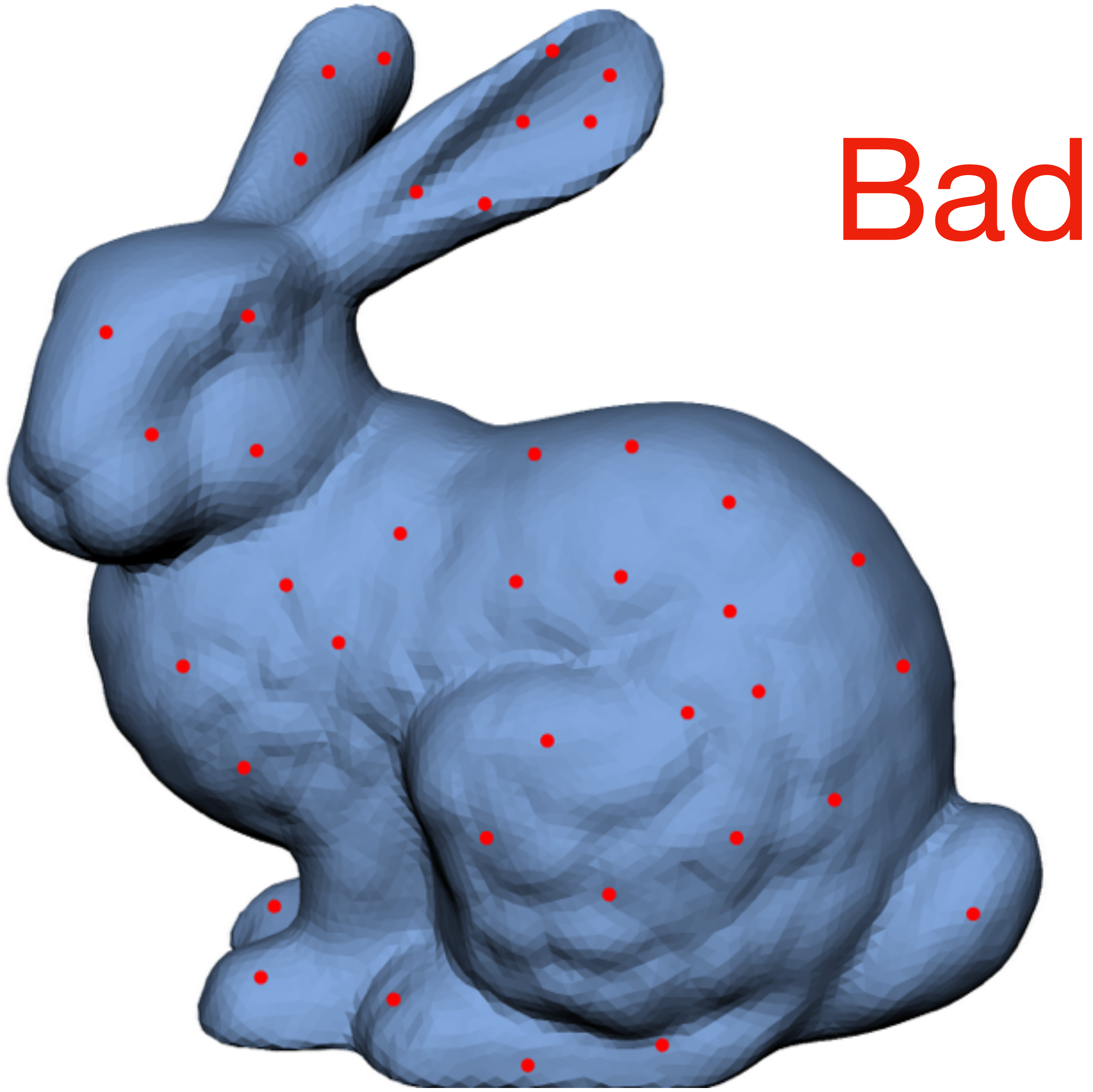

Limitations

- Only works if the requested values are local.
- The local problem needs to be formulated in terms of a global operator.

Conclusion

- Back-substitution can be significantly accelerated if only a few values are requested.
- If requested values belong to a localized patch on a mesh, performance benefits the most (because of nested dissection).
- Considering sparsity in the solution can improve performance by orders of magnitude.
- The method is trivial to implement.

Thank you for your attention!

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Applications: Auto-diffusion

- Compute heat diffusion from a single source vertex and evaluate the amount of heat staying at the source after some time.

- Only one value in $\mathbf{x}$ is required.